# From $\beta$-Expansions to Chaos and Fractals

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**Abstract.**

In this paper we will study the behaviour of the following difference equation: $x_{k+1} = \{\beta x_k\}$, where $\beta$ may be an integer, a real, or even a complex or a matrix, and $\{.\}$ is the fractional part; depending on $\beta$ and $x_0$, the nature of the solution $x_k$ changes dramatically and chaotic behaviour occurs; by studying different properties of the set of all $x_k$, we look for eventual cycles, density results, and discuss Lebesgue measure, category and fractal dimension problems.

This equation occurs in applied mathematics, for instance in control theory when one considers a single input linear time-invariant system controlled by a particular state feedback control. Well-known fractal sets can also be deduced from that equation for particular values of $\beta$.

Many questions related to this difference equation are still open problems in number theory and in spite of its apparent simplicity, this equation turns out to be a real challenge.

## 1. Introduction

Chaos is a very popular theme of research nowadays; it occurs "naturally" in many complex systems. In our case, we were dealing with a control problem: given some class $\mathcal{S}$ of systems with inputs and outputs, given a family of control laws (i.e. a function giving the input as a function of time, depending eventually upon the previous inputs and outputs), is it possible for any system in $\mathcal{S}$ to find a control law that stabilizes that system either to an equilibrium point or around a reference signal?

Let us take $\mathcal{C}$ as the class of discrete single-input controllable time-invariant linear systems given by their state equation: $X_{k+1} = AX_k + Bu_k$, where $X_{k+1}$ and $X_k$ are $n$-dimensional vectors (the state vectors), $A$ is an $(n \times n)$ matrix (the state matrix), $B$ is an $n$-dimensional vector (the control matrix) and $u_k$ is a scalar (the control law); the considered control laws are incremental control laws ($u_{k+1} = u_k + e_k\Delta$, where $\Delta$ is a positive real and $e_k$ an integer between $-m$ and $+m$); the goal is to find some $u_k$ as a function of $X_k$, $A$ and $B$ that stabilizes the system, i.e. keeps $X_k$ bounded. The whole discussion and the corresponding expression of $u_k$ can be found in [1]. If the state space is monodimensional ($n = 1$), then $B = 1$ and $u_k$ can be shown to be rewritten as $u_k = -[AX_k]$, hence $X_{k+1} = \{AX_k\}$.

For state spaces with dimension $n = 2$, we simulated on a computer the trajectory of the state vector and as can be seen in Figure 1, the trajectory differs considerably depending on the values of $A$, going from limit cycles to apparent density. This was interesting for us, because while our choice of $u_k$ guaranteed that the system was stable (i.e. the state vector was uniformly bounded), that law implied a "chattering" of the

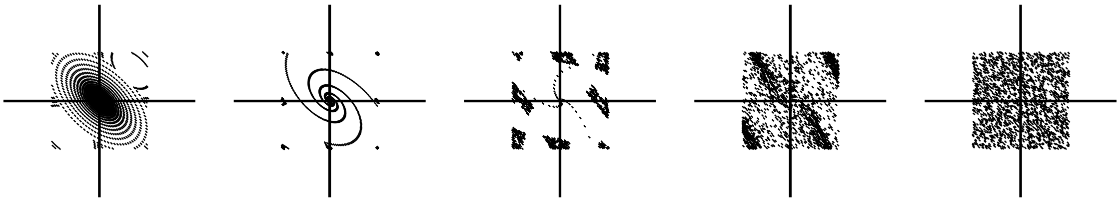

Figure 1: Trajectories of different 2-dimensional systems.

state vector inside the bounds which seemed to have very different shapes for different state matrices. The study of the monodimensional case, as exposed here, has given us much insight on the chaotic behaviour of the system controlled by the control law $u_k$.

Let us recall some notations before we enter the heart of the study. For any positive real $x$ we consider its decimal expansion:

$$x = \sum_{i=0}^{p} d_i 10^i + \sum_{i=-\infty}^{-1} d_i 10^i$$

The left sum is an integer, called the *integer part* and denoted by $[x]$, while the right sum is a real in $[0,1[$, called the *fractional part* and denoted by $\{x\}$.

The paper deals with the difference equation: $x_{k+1} = \{\beta x_k\}$, where $\beta$ can be an integer, a real, a complex, or even a matrix, if we extend in a natural way the fractional part from reals to vectors (by considering actually the various components of the vector). It is obvious that $x_k$ takes its values in the interval $[0,1[$ as soon as $k \geq 1$; we will assume $x_0 \in [0,1[$ too. What will interest us most is the behaviour of the set of all $x_k$ for a given $x_0$, in other words what properties has the orbit of $x_0$ under the operator $T : x \mapsto \{\beta x\}$.

We will first consider integer values of $\beta$ and show how that leads to chaos; then we will extend $\beta$ to real and complex values, relating then our problem to some more or less known fractal curves.

## 2. $\beta$ integer

Of course $\beta$ will be assumed greater than or equal to 2, as the cases $\beta = 0$ and $\beta = 1$ are trivial.

Let us write the difference equation differently: instead of writing $x_{k+1} = \{\beta x_k\}$, we can define an integer $q_{k+1}$ such that $\beta x_k = q_{k+1} + x_{k+1}$. By induction, we have then:

$$x_0 = \frac{q_1}{\beta} + \cdots + \frac{q_k}{\beta^k} + \frac{x_k}{\beta^k}$$

This formula yields an expansion of $x_0$ in base $\beta$, called *$\beta$-expansion*, where $x_k$ is the remainder in the $\beta$-expansion of $x_0$ after having isolated the $k$ first digits. The map $T$ is then a shift operator, as $x_{k+1}$ is obtained from $x_k$ but shifting the $\beta$-expansion of $x_k$ one step to the left and losing in this way the leading digit.

If $\beta$ is an integer, the $\beta$-expansion exists for any $x_0 \in [0,1[$ and is unique (as is the case in the decimal notation, we reject all expansions for which all digits from a given rank are equal to $\beta - 1$). The unicity comes from the fact that $\sum_{i=1}^{\infty} a_i \beta^{-i} = \sum_{i=1}^{\infty} b_i \beta^{-i}$ implies $a_1 - b_1 = \sum_{i=1}^{\infty} (b_{i+1} - a_{i+1}) \beta^{-i}$, and the term on the right side is smaller than or equal to $\sum_{i=1}^{\infty} (\beta - 1) \beta^{-i} = 1$, where the equality only takes place when all $b_i$, $i \geq 2$, are equal to $\beta - 1$ and all $a_i$, $i \geq 2$, are null, which is impossible, as former expansions have been rejected by assumption. As $a_1 - b_1$ is an integer, it has to be equal to 0; by induction, we prove then that $a_i = b_i$ in the same way.

Looking at the orbit of a given $x_0$ under the operation of $T$, i.e. looking at the successive remainders in the $\beta$-expansion of $x_0$, raises open questions: for instance for $x_0 = \pi$ and $\beta = 10$, it is not known whether this orbit is finite or not. We will (alas!) not give an answer to that question, and will therefore not consider the orbit of a particular real and try to study its properties; instead we will consider the set of reals whose orbit has some given property and try to characterize that set.

We introduce the sets $P$ and $D$ (and their complements $P^c$ and $D^c$):

$$\begin{aligned} P &= \{x \in [0,1[ \mid \text{the orbit of } x \text{ under } T \text{ is finite}\} \\ D &= \{x \in [0,1[ \mid \text{the orbit of } x \text{ under } T \text{ is dense in } [0,1]\} \end{aligned}$$

We note that actually, an orbit is finite if and only if it is periodic, i.e.:

$$\exists \tau, \exists n, \forall k \geq n, x_{k+\tau} = x_k$$

$\tau$ is called the *period* of that orbit. It should be noticed that a periodic orbit consists on the one hand of a finite number of points which may be visited only once, and on the other hand of a set of points which are visited periodically every $\tau$ steps. Indeed if the orbit of $x_0$ is finite there must be some $x$ in the orbit of $x_0$ which is visited at least twice; as to any element of $[0,1[$ corresponds in a unique way a $\beta$-expansion, and as the $n$th iterate of $T$ applied to $x$ shifts its expansion $n$ times to the left, this means that $x$ contains in its own expansion a copy of itself starting from some rank, which implies this expansion is ultimately periodic: the $\beta$-expansion of $x_0$ starts with a finite string of digits followed by a periodic expansion (another finite string repeating itself ad infinitum). Conversely an ultimately periodic expansion implies a periodic orbit. Hence, $P = [0,1[ \cap \mathbf{Q}$.

In particular, the set of all $x \in P$ of period 1 is the set of $x$ which have an ultimately periodic $\beta$-expansion with period 1 (this includes the $x$ which have a finite $\beta$-expansion). This set is of relative importance for our control application, as it represents the set of state vectors which generate a trajectory converging to a point. *It is dense in* $[0,1]$.

As $P$ is only countable while $[0,1[$ is uncountable, there exist infinite orbits; the natural question is then to ask how these orbits can be distributed inside $[0,1]$. There are many ways to define the "distribution" of reals inside an interval, and this is a major issue when defining the efficiency of a random generator for instance (for a discussion on the subject, one may refer to [2]). We will consider only equidistribution and density.

Let us recall that a sequence $(x_n)_{n\in\mathbf{N}}$ is *equidistributed* [3] or *uniformly distributed modulo 1* if for all $a, b$ in $[0,1[$, where $a < b$, we have:

$$\lim_{N\to\infty} \sum_{n=1}^{N} \frac{\mathbf{1}_{[a,b[}(x_n)}{N} = b - a,$$

where $\mathbf{1}_{[a,b[}$ is the characteristic function of the interval $[a,b[$.

In [4] it is shown that the sequence $x_{n+1} = \{\beta x_n + \theta\}$, where $\beta$ is a fixed integer greater than or equal to 2 and $\theta$ a fixed real, is equidistributed for almost all $x_0$ and for all $\theta$; taking $\theta = 0$, we conclude that *for almost all $x_0$ the orbit of $x_0$ is equidistributed.* For $\beta = 1$, the sequence $x_{n+1} = \{\beta x_n + \theta\}$ is known to be equidistributed for all $x_0$ and only for all irrational $\theta$. Of course in our case, $\theta = 0$, so the orbit of any $x_0$ is not equidistributed, but it was obvious since it is reduced to a point.... It should be noted that although the orbit of $x_0$ is equidistributed for almost all $x_0$ for $\beta \geq 2$, the transcendence of $x_0$ is not a sufficient condition for equidistribution: take $x_0 = \sum_{\nu=1}^{\infty} N^{-\nu!}$, the Liouville number, which is known to be transcendental; then for $n \geq 3$, $x_n < N^{-1}+N^{-2}+N^{-3} < 1$ and the sequence $(x_n)$ cannot be thus equidistributed in $[0,1[$.

The next proposition will characterize the set of $x$ which have a dense orbit and the set of $x$ which do not have a dense orbit, in terms of countability, Lebesgue measure and category.

We will need to consider $[0,1]$ as a topological space; if the finite set $\{0,\cdots,\beta-1\}$ has the discrete topology, we can consider on $[0,1] = \{0,\cdots,\beta-1\}^{\mathbf{N}}$ the product topology [5]. The $O[b_1\cdots b_k]$, containing all $\beta$-expansions starting with $b_1\cdots b_k$ followed by anything, are a basis for this product topology. Considering this topology is natural as we have rejected redundant expansions (those containing an infinity of $\beta-1$).

**Proposition 1** *For integer values of $\beta$, we have the following results:*

1. *$D$ and $D^c$ are uncountable*
2. *$D$ and $D^c$ are dense in $[0,1]$*
3. *$D$ has Lebesgue measure 1 (and of course $D^c$ has Lebesgue measure 0)*
4. *$D^c$ is first category in $[0,1]$ (and of course $D$ is second category)*

*Proof:* The proof will be made for $\beta = 2$ for sake of simplicity, but the same methods of proof hold for any integer $\beta$, mutatis mutandis.

*D is not empty*

Let $x$ be given by its binary expansion:

$$x = 0.0\ 1\ \ 00\ 01\ 10\ 11\ \ 000\ 001\ 010\ 011\ 100\ 101\ 110\ 111\ \ldots,$$

in other words, the binary expansion contains every finite sequence of 0's and 1's; then the orbit of $x$ visits every neighbourhood in $]0,1[$ infinitely often and is therefore dense; the same holds of course for any $x$ which has all finite binary sequences as substrings in its binary representation arranged in any order; conversely, if the orbit of $x$ is dense in $]0,1[$, then some point in its orbit must be arbitrarily close to any $0.b_1b_2\cdots b_k$ (any finite binary expansion in $]0,1[$) and hence this point must begin its binary expansion with the same string of 0's and 1's; hence $x$ contains any finite string of 0's and 1's. This characterizes $D$. Of course $D^c$ is the set of all $x$ whose expansion lacks some finite string of 0's and 1's.

*D is uncountable*

Let:

$$x = 0.0\ 1\ \ d_1\ \ 00\ 01\ 10\ 11\ \ d_2\ \ 000\ 001\ 010\ 011\ 100\ 101\ 110\ 111\ \ldots,$$

in other words $x$ contains every infinite sequence of 0's and 1's written in increasing lexicographical order, and between blocks of substrings of different length we insert a digit $d_i$. It is then obvious to define an injective map from $[0,1[$ (to any real $y$ corresponds the $x$ built in this way where the digits $d_i$ are the successive digits of the binary expansion of $y$). The uncountability follows.

*$D^c$ is uncountable*

We will prove that the set of all reals which do not have 11 in their binary expansion is an uncountable subset of $D^c$. Let $y$ be in $]0,1[$ and let us define $f$ in the following way. The binary expansion of $y$ can be seen as a succession of blocks of 1's and 0's; the first block has length $n_1$, the second has length $n_2$, and so on. The binary expansion of $y$ starts either with a block of 1's or with a block of 0's. In the first case the binary expansion of $f(y)$ will start with a 1 followed by $n_1$ 0's, then a 1, then $n_2$ 0's, and so on, while in the second case the binary expansion of $f(y)$ starts with $n_1$ 0's, then a 1, then $n_2$ 0's, and so on. To summarize ($1^n$ denotes a block of $n$ 1's; e.g. .101 is the real $2^{-1}+2^{-3}$):

$$\begin{aligned} y = .1^{n_1}\,0^{n_2}\ldots &\mapsto f(y) = .1\,0^{n_1}\,1\,0^{n_2}\,1\ldots \\ y = .0^{n_1}\,1^{n_2}\ldots &\mapsto f(y) = .0^{n_1}\,1\,0^{n_2}\,1\ldots \end{aligned}$$

$f$ is obviously a bijective map (injectivity uses the unicity of the binary expansion), hence $D^c$, which has an uncountable subset, is uncountable.

*$D$ and $D^c$ are dense in* $[0,1]$

Let $y$ be in $[0,1]$, let $\epsilon$ be the approximation error; for $k$ such that $2^{k+1} \leq \epsilon$, let $b_1 \cdots b_k$ be the binary expansion of $y$ up to rank $k$. Let $x$ be any member of $D$ (respectively $D^c$); then the number whose binary expansion begins with $b_1 \cdots b_k$ and goes on with the expansion of $x$, is in $D$ (respectively in $D^c$), because in this way we only added a finite number of reals to its orbit, which does not change density or non-density. Furthermore this number is at a distance smaller than $\epsilon$ from $y$. We conclude that $D$ and $D^c$ are dense in $[0,1]$.

*$D^c$ has Lebesgue measure* 0

Let $s$ be a finite string of 0's and 1's. We will show that the set of $x$ which have a finite binary expansion, where $s$ does not appear, has measure 0. Suppose $s$ has length $n$. The number of possible binary strings of length $nN$ in which $s$ does not appear is at most $(2^n-1)^N$, since there are in each block of length $n$ only $2^n-1$ strings different from $s$ (the number of such possible strings is smaller because we do not consider overlapping possibilities of $s$ between the different blocks). Thus the measure of the set of $x$ such that $s$ does not appear within the first $nN$ bits is smaller than $(2^n-1)^N/2^{nN} = (1-2^{-n})^N$. As $N$ goes to infinity, this number goes to 0, as $1-2^{-n} < 1$. The set of all $x$ which have a binary expansion in which $s$ does not appear is contained in all these sets, whose measure becomes arbitrarily small, so it itself has measure zero. As there are countably many strings $s$ and the union of a countable collection of sets of measure zero has measure zero, and as $D^c$ is the union of the sets of $x$ which lack a string $s$ in their binary expansion, we conclude that $D^c$ has measure 0. Of course, *$D$ has Lebesgue measure* 1.

*$D^c$ is first category in* $[0,1]$ *for the product topology*

As we noted before, if we call $S[b_1 \cdots b_k]$ the set of all $x$ whose binary expansion has no occurrence of the substring $b_1 \cdots b_k$, then $D^c$ is the union of all such sets; as the number of such sets is countable, we only have to prove any $S[b_1 \cdots b_k]$ is nowhere dense, i.e. its adherence has empty interior. For the product topology, $S[b_1 \cdots b_k]$ is closed: let us assume it has some real $r$ with $b_1 \cdots b_k$ in its $\beta$-expansion as a limit point; then for any open set containing that real $r$, there must be an infinity of elements of $S[b_1 \cdots b_k]$, but this is obviously impossible because any such open set is an $O[a_1 \cdots a_n]$ where $a_1 \cdots a_n$ is the beginning of the expansion of $r$, and *has* to include for large $n$ the string $b_1 \cdots b_k$. Let us now assume $S[b_1 \cdots b_k]$ has a non-empty interior for that product topology, and let then $]a, b[$ be an open interval included in some $S[b_1 \cdots b_k]$; there exists necessarily a number in $]a, b[$ which has the substring $b_1 \cdots b_k$ in its binary expansion (as the numbers which have $b_1 \cdots b_k$ in their binary expansion are dense in $[0, 1]$ for the product topology), which contradicts the fact that $]a, b[$ is included in $S[b_1 \cdots b_k]$. We conclude that $D^c$ is of first category in $[0, 1]$, and as the union of two sets of first category is first category and $[0, 1]$ is of second category in itself, we conclude that *$D$ is of second category in* $[0, 1]$ for the product topology.

□

Thus the set of all $x$ whose orbit is dense is a "large" set of $[0, 1]$ in terms of measure and category. On the other hand, the set of all $x$ whose orbit is non-dense is a "small" set of $[0, 1]$ in these terms. However both these sets are uncountable and dense.

If we consider other topologies on $[0, 1]$ like the usual order topology, the proof characterizing the category of $D^c$ does not hold any longer: let us take for instance $\beta = 10$, then $S[89]$ is closed, but $S[56]$ is not closed (e.g. 0.89 can only be approximated by reals which have 89 in their decimal expansion, either reals like 0.890001 or like 0.88999; on the other hand the reals 0.55999 with increasing number of 9 form a sequence in $S[56]$ converging to 0.56). It is easy to see for which $b_1 \cdots b_k$ the corresponding set $S[b_1 \cdots b_k]$ is closed; that set has empty interior (the proof of the previous proposition is valid). It is obvious that non-closed $S[b_1 \cdots b_k]$ are dense. However, the properties of nowhere density and first category are invariant for the product topology and the usual order topology [6], so that $D^c$ is first category for the usual order topology too and $D$ is second category.

To see the sensitivity of that difference equation, let us take two arbitrarily close numbers which differ only at the $k$th step; after $k$ iterations, the distance between these two numbers will have been multiplied by $\beta^k$, which is an exponential function of $k$ with exponential rate $\ln \beta$, called the *Lyapunov exponent* of $T$. The sensitivity of that equation, and the fact that rational $x_0$ have finite orbits, makes every attempt to draw the orbit of $x$ as a function of $x$ an impossible task. Although we did not make a similar study of the bidimensional case, we are pretty sure the same phenomena should occur and Figure 1 has to be taken with the greatest care.

All the previous results hold for $\beta$ negative too. For negative $x$, we define $[x] = -[-x]$ and $\{x\} = -\{-x\}$. It should be noted that the integer part is *not* the largest integer smaller than $x$ as it is usually the case; any of the two definitions would lead to the same results we are interested in, but with our definition, the symmetry between positive and negative $x$ is not broken.

For negative $\beta$, $T(x_0)$ is obtained from $x_0$ by shifting its expansion to the left and multiplying by $-1$; thus the iterates of $T$ work as if $\beta$ were positive, but the remainders are alternatively in $]-1, 0]$ and $[0, 1[$; it is easily seen that any $x$ for which the orbit ultimately reduces to a point must have a finite expansion, as 0 is the only value belonging to $]-1, 0]$ and $[0, 1[$; the set of such $x$ is countable and dense in $[0, 1]$. Periodic expansions with period 1 lead to orbits with two points with same absolute value. An orbit is periodic if and only if the expansion of $x$ is periodic. Finally an orbit is dense if and only if $x$ has all finite strings in its expansion: just note that if an expansion has all strings, then it has $b_1 \cdots b_k b_1 \cdots b_k$ and $b_1 \cdots b_k 0 b_1 \cdots b_k$ for instance, which means that the set of the expansions with all strings is equal to the set of the expansions with all strings starting at odd ranks (respectively even ranks) in the corresponding expansion.

## 3. Extensions to $\beta$ real, complex; relations to fractals

If $\beta$ is in $]-1,1[$, $T$ is a contracting map, and the orbit converges to a limit point, namely 0.

When $\beta$ takes real values greater than 1, things get much more difficult than in the previous section, because unicity of the $\beta$-expansion is lost. It is easily seen that a $\beta$-expansion exists always for all $x \in [0,1]$, namely the expansion yielded by iterative application of $T$. But this time, the integers $q_i$ defined as above ($q_i = [\beta x_i]$) take values in $\{0, \cdots, [\beta]\}$ and as $\sum_{i=1}^{\infty} [\beta]\beta^{-i}$ equals $[\beta]/(\beta-1) > 1$, we already conclude that some expansions do not correspond to $x \in [0,1]$ and cannot be considered. Furthermore, if $\sum_{i=1}^{\infty} c_i \beta^{-i}$ is an expansion of 1, and we take $a_1 = b_1 + 1$ and $b_i - a_i = c_i$ for $i \geq 2$, we have $\sum_{i=1}^{\infty} a_i \beta^{-i} = \sum_{i=1}^{\infty} b_i \beta^{-i}$; hence the loss of unicity.

Given an expansion, there may be no $x$ in $[0,1]$ corresponding to that expansion, or there may be several expansions leading to the same $x$, in which case the properties of the expansion do not characterize the given $x$, unless we are sure that expansion has been obtained by iterative application of $T$. To summarize, we cannot use the same methods as in the case $\beta$ integer.

Of course, we can give trivial characterizations: for instance, any $x$ such that its orbit is periodic has a periodic expansion (among the different expansions), and hence the set of $x$ which have a periodic orbit is at most countable. But that does not characterize such $x$ in a satisfactory manner (like in the case $\beta$ integer, where $x$ has a periodic orbit if and only if $x$ is rational). When $\beta$ is real, only sufficient conditions are known [7] (other known facts and open questions can be found in this reference).

Even for fractional $\beta$, we can see the situation is very different from the integer case: *for $\beta$ rational non-integer, 1 has an infinite aperiodic $\beta$-expansion.*

*Proof:* Let us write $\beta$ under its irreducible form $p/q$, where $p$ and $q$ are two integers with no common divisors. Let us assume 1 has a periodic $\beta$-expansion from some point:

$$1 = b_1\beta^{-1} + \cdots + b_m\beta^{-m} + \sum_{i=0}^{\infty}(a_1\beta^{-m-1-i} + \cdots + a_n\beta^{-m-n-i})$$

The finite term with the $b_i$'s comes from the fact that the periodic expansion does not have to start from scratch. Multiplying by $\beta^{m+n}$, we have:

$$\beta^{m+n} = b_1\beta^{m+n-1} + \cdots + b_m\beta^{n} + a_1\beta^{n-1} + \cdots + a_n + 1 - b_1\beta^{-1} + \cdots + b_m\beta^{-m}$$

because the periodic part of the expansion is invariant under $n$ shifts. Writing then $p/q$ instead of $\beta$ and getting rid of any negative power, we have:

$$\begin{aligned} p^{2m+n} &= b_1p^{2m+n-1}q + \cdots + b_mp^{m+n}q^m \\ &\quad + a_1p^{m+n-1}q^{m+1} + \cdots + a_np^mq^{m+n} + p^mq^{m+n} \\ &\quad - b_1p^{m-1}q^{m+n+1} - \cdots - b_mq^{2m+n} \end{aligned}$$

This expression contains only integers; as $p$ divides the left side of the equality, it must divide the right side too, and as $p$ and $q$ are relatively prime, $p$ divides $b_m$; but $b_m$ is a digit of the expansion, so that $b_m \leq [\beta] = [p/q] < p$ because $q \geq 2$ as $\beta$ is not an integer. This implies $b_m = 0$; an obvious induction (based on the fact that in this expression the powers of $p$ are decreasing) yields then $b_1 = \ldots = b_m = 0$. The equality can then be simplified by $p^m$ and it is easily seen that then $p$ has to divide $(a_n+1)q^{m+n}$; but $\beta$ is rational, non-integer and greater than 1, so that $p \geq 3$ and $q \geq 2$;

then $a_n + 1 \leq [p/q] + 1 < p/q + 1 \leq p/2 + 1 < p$. This implies $a_n = 0$; an obvious induction yields then $a_1 = \ldots = a_n = 0$, which is contradictory. $\square$

By writing the condition for a periodic $\beta$-expansion of any $x_0$, it is easy to see that if $x_0$ is rational, then $\beta$ has to be algebraic. But it is not sufficient, as we have just seen.

Concerning eventual measure questions, there is an important result: for all real $\beta$, *$T$ is ergodic* (any Lebesgue measurable subset of $[0,1]$ invariant by this map with non-null Lebesgue measure has necessarily measure 1) [8, 9]. Actually the proof of the equidistribution of $x_n$ for integer $\beta$ in [4] makes use of the ergodicity of $T$.

Let us define the *similarity dimension* of a set $S$ by: if $T_i$ are contractions with scale factors $r_i < 1$ such that $S = \bigcup_i T_i(S)$, then the similarity dimension $s$ satisfies the equation: $\sum_i r_i^s = 1$.

Concerning $\beta$-expansions, we have then: the set of all $\beta$-expansions has similarity dimension $\ln([\beta]+1)/\ln\beta$. This can actually be generalized in the following manner: the set of all expansions $F = \{\sum_{i=1}^{\infty} a_i \beta^{-i}\}$ where all $a_i$ are integers in $\{d_1, \ldots, d_k\}$ has similarity dimension $\ln k/\ln\beta$ ($T$ performs $k$ copies of the set of all $\beta$-expansions scaled by factor $1/\beta$, hence the equation $k(1/\beta)^s = 1$).

In the literature, some particular values of $\beta$ have been studied; in [2, 10], some fractional and some complex values have led to fractal sets. For instance the set of all expansions $F$ associated to $\beta = -1 + i$ and $d_1 = 0, d_2 = 1$ yields a fractal made of two adjacent Heighway dragons; if positive powers of $\beta$ are allowed, it can be shown that every complex number can be represented in this number system, whence the whole plane is covered with countably many such dragons which furthermore can be shown to overlap only in their boundaries. Other fractal sets have been investigated, for instance for $\beta = -2$ and $d_1 = 0, d_1 = 1, d_3 = \omega, d_4 = \omega^2$ where $\omega = e^{2i\pi/3}$).

The last extension will concern matrices: if $\beta$ is a matrix and $X_k$ a vector, we define $\{X_k\}$ by the vector whose components are the fractional part of every component of $X_k$. We can then work on the difference equation: $X_{k+1} = \{\beta X_k\}$. As is suggested by Figure 1, this seems to lead to chaotic behaviour too. We did not study yet that issue, but can already state the next results which are logical generalizations of the monodimensional case.

The proof for the next two results can be found in [12].

The operator $T$ which maps $X$ onto $\{\beta X\}$ is *ergodic* if and only if the matrix $\beta$ is non-singular and none of its eigenvalues is a root of unity.

The sequence $X_{k+1} = \{\beta X_k\}$ is *equidistributed* (immediate generalization of the concept of equidistribution) for almost all $X_0$ if and only if $\beta$ is non-singular and none of its eigenvalues is a root of unity.

In [12], the more general case of the transformation $X \mapsto \{\beta X + \Theta\}$ is studied. It is worthy to note that the equivalence holds only for $\Theta = 0$, which is luckily the case we are interested in.

## 4. Concluding remarks

In this paper we have studied a difference equation, which in spite of an apparent simplicity, generates very complex trajectories and raises quite a few questions. Chaos has been identified in the solutions of that equation in various ways.

In the easiest case ($\beta$ integer), we have tried to characterize these trajectories topologically, and seen the relations with some open problems in number theory. It appeared then natural to extend the investigation; this raises many other open problems in number theory and symbolic dynamics, and a link with well-known fractal curves has been made.

This study is all the more interesting since the equation occurs in real situations, one of which is the control of a dynamical system with a particular input control law. Further research could focus on some of the presented extensions, like the complex or the matricial case: characterizing topologically the trajectories in the matricial case could lead to a better understanding of the generation of some fractals.

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