Discrete mathematics with R: introducing the permutations package

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Abstract

Here I introduce the **permutations** package, for manipulating and permutations of a finite set.

Keywords: Permutations.

1. Overview

Permutations of a finite set are an important and interesting branch of mathematics, having links in combinatorics (Stanley 2011), group theory (Milne 2013), and various branches of recreational mathematics (Averbach and Chein 2000).

2. Introduction

We consider bijections from the set $[n] = \{1, 2, 3, \ldots, n\}$ to itself. For a specific example, take $f\colon [9] \longrightarrow [9]$ defined by the following diagram:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 9 & 2 & 6 & 3 & 5 & 4 & 1 & 7 & 8 \end{pmatrix}$$

Thus $f(1) = 9$, $f(2) = 2$, and so on. Function $f(\cdot)$ may be inverted by inspection:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 7 & 2 & 4 & 6 & 5 & 3 & 8 & 9 & 1 \end{pmatrix}$$

If we wish to determine, say, $f(f(f(\cdot))) = f^3(\cdot)$, then it is more convenient to represent f in *cycle form*:

$$(1987)(346)$$

which is a compact representation of the fact that $1 \xrightarrow{f} 9 \xrightarrow{f} 8 \xrightarrow{f} 7 \xrightarrow{f} 1$ and $3 \xrightarrow{f} 4 \xrightarrow{f} 6 \xrightarrow{f} 3$, the remaining elements mapping to themselves. Then it is clear that f^3 is the cycle (1789).

The R idiom for the above would be:

```
> library("permutations")
> f <- as.word(c(9,2,6,3,5,4,1,7,8))
> f

    {1} {2} {3} {4} {5} {6} {7} {8} {9}
[1] 9   .   6   3   .   4   1   7   8

> inverse(f)

    {1} {2} {3} {4} {5} {6} {7} {8} {9}
[1] 7   .   4   6   .   3   8   9   1

> as.cycle(f)

[1] (1987)(364)

> f^3

[1] (1789)
```

(a dot in the word form indicates that the element in question is mapped to itself). Given another such cycle $g = (142)$ then we may combine f and g in two ways with $fg = (19872364)$ and $gf = (12643987)$. In R:

```
> g <- as.cycle(c(1,4,2))
> f*g

    {1} {2} {3} {4} {5} {6} {7} {8} {9}
[1] 9   1   6   3   .   2   4   7   8

> g*f

    {1} {2} {3} {4} {5} {6} {7} {8} {9}
[1] 3   9   6   2   .   4   1   7   8
```

One measure of the non-commutativity of f and g is the *commutator*, here defined as $f^{-1}g^{-1}fg$:

```
> commutator(f,g)

    {1} {2} {3} {4} {5} {6} {7} {8} {9}
[1] 4   3   9   2   .   .   .   .   1
```

The package is vectorized. Suppose we wish to consider the symmetry group of an icosahedron, known to be the even permutations of a set of five elements:

```
> S5 <- as.cycle(t(partitions::perms(5)))
> A5 <- S5[is.even(S5)]
> A5

 [1] ()         (345)     (354)     (23)(45) (234)     (235)     (243)     (245)
 [9] (24)(35) (253)     (254)     (25)(34) (12)(45) (12)(34) (12)(35) (123)
[17] (12345)  (12354)  (12453)  (124)     (12435)  (12543)  (125)     (12534)
[25] (132)     (13452)  (13542)  (13)(45) (134)     (135)     (13)(24) (13245)
[33] (13524)  (13)(25) (13254)  (13425)  (14532)  (142)     (14352)  (143)
[41] (145)     (14)(35) (14523)  (14)(23) (14235)  (14253)  (14325)  (14)(25)
[49] (15432)  (152)     (15342)  (153)     (154)     (15)(34) (15423)  (15)(23)
[57] (15234)  (15243)  (15324)  (15)(24)
```

where function `perms()` is taken from the **partitions** package (Hankin 2006). Thus `S5` is all permutations of size 5, and `A5` just the even permutations. We might consider the first four elements of vector `A5`, and combine with a cycle of length 9:

```
> A5[1:4]*cyc_len(9)

    {1} {2} {3} {4} {5} {6} {7} {8} {9}
[1] 2   3   4   5   6   7   8   9   1
[2] 2   3   5   6   4   7   8   9   1
[3] 2   3   6   .   .   7   8   9   1
[4] 2   4   .   6   .   7   8   9   1
```

As a final illustration, we may calculate the conjugate[1] the of the four permutations with another element:

```
> A5[1:4]^as.word(7:1)

    {1} {2} {3} {4} {5} {6} {7}
[1] .   .   .   .   .   .   .
[2] .   .   5   3   4   .   .
[3] .   .   4   5   3   .   .
[4] .   .   4   3   6   5   .

> as.cycle(A5[1:4]^as.word(7:1))

[1] ()       (354)    (345)    (34)(56)
```

References

[1] The conjugate of x and y, written x^y, is defined as $y^{-1}xy$; the notation is motivated by the fact that $x^z y^z = (xy)^z$ and $x^{yx} = (x^y)^z$

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