Credibility theory features of **actuar**

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1 Introduction

Credibility models are actuarial tools to distribute premiums fairly among a heterogeneous group of policyholders (henceforth called *entities*). More generally, they can be seen as prediction methods applicable in any setting where repeated measures are made for subjects with different risk levels.

The credibility theory features of **actuar** consist of matrix `hachemeister` containing the famous data set of Hachemeister (1975) and function `cm` to fit hierarchical (including Bühlmann, Bühlmann-Straub) and regression credibility models. Furthermore, function `simul` can simulate portfolios of data satisfying the assumptions of the aforementioned credibility models; see the `"simulation"` vignette for details.

2 Hachemeister data set

The data set of Hachemeister (1975) consists of private passenger bodily injury insurance average claim amounts, and the corresponding number of claims, for five U.S. states over 12 quarters between July 1970 and June 1973. The data set is included in the package in the form of a matrix with 5 rows and 25 columns. The first column contains a state index, columns 2–13 contain the claim averages and columns 14–25 contain the claim numbers:

```
\begin{Sinput}
> data(hachemeister)
> hachemeister
\end{Sinput}
\begin{Soutput}
     state ratio.1 ratio.2 ratio.3 ratio.4 ratio.5 ratio.6
[1,]     1    1738    1642    1794    2051    2079    2234
[2,]     2    1364    1408    1597    1444    1342    1675
[3,]     3    1759    1685    1479    1763    1674    2103
[4,]     4    1223    1146    1010    1257    1426    1532
[5,]     5    1456    1499    1609    1741    1482    1572
     ratio.7 ratio.8 ratio.9 ratio.10 ratio.11 ratio.12
[1,]    2032    2035    2115     2262     2267     2517
[2,]    1470    1448    1464     1831     1612     1471
[3,]    1502    1622    1828     2155     2233     2059
[4,]    1953    1123    1343     1243     1762     1306
[5,]    1606    1735    1607     1573     1613     1690
     weight.1 weight.2 weight.3 weight.4 weight.5 weight.6
[1,]     7861     9251     8706     8575     7917     8263
[2,]     1622     1742     1523     1515     1622     1602
[3,]     1147     1357     1329     1204      998     1077
[4,]      407      396      348      341      315      328
[5,]     2902     3172     3046     3068     2693     2910
     weight.7 weight.8 weight.9 weight.10 weight.11
[1,]     9456     8003     7365      7832      7849
[2,]     1964     1515     1527      1748      1654
[3,]     1277     1218      896      1003      1108
[4,]      352      331      287       384       321
[5,]     3275     2697     2663      3017      3242
     weight.12
[1,]      9077
[2,]      1861
[3,]      1121
[4,]       342
[5,]      3425
\end{Soutput}
```

3 Hierarchical credibility model

The linear model fitting function of R is named `lm`. Since credibility models are very close in many respects to linear models, and since the credibility model fitting function of **actuar** borrows much of its interface from `lm`, we named the credibility function `cm`.

Function `cm` acts as a unified interface for all credibility models supported by the package. Currently, these are the unidimensional models of Bühlmann

(1969) and Bühlmann and Straub (1970), the hierarchical model of Jewell (1975) (of which the first two are special cases) and the regression model of Hachemeister (1975), optionally with the intercept at the barycenter of time (Bühlmann and Gisler, 2005, Section 8.4). The modular design of `cm` makes it easy to add new models if desired.

This subsection concentrates on usage of `cm` for hierarchical models.

There are some variations in the formulas of the hierarchical model in the literature. We compute the credibility premiums as given in Bühlmann and Jewell (1987) or Bühlmann and Gisler (2005). We support three types of estimators of the between variance structure parameters: the unbiased estimators of Bühlmann and Gisler (2005) (the default), the slightly different version of Ohlsson (2005) and the iterative pseudo-estimators as found in Goovaerts and Hoogstad (1987) or Goulet (1998).

Consider an insurance portfolio where contracts are classified into cohorts. In our terminology, this is a two-level hierarchical classification structure. The observations are claim amounts S_{ijt}, where index $i = 1, \dots, I$ identifies the cohort, index $j = 1, \dots, J_i$ identifies the contract within the cohort and index $t = 1, \dots, n_{ij}$ identifies the period (usually a year). To each data point corresponds a weight — or volume – w_{ijt}. Then, the best linear prediction for the next period outcome of a contract based on ratios $X_{ijt} = S_{ijt}/w_{ijt}$ is

$$\begin{aligned} \hat{\pi}_{ij} &= z_{ij} X_{ijw} + (1 - z_{ij}) \hat{\pi}_i \\ \hat{\pi}_i &= z_i X_{izw} + (1 - z_i) m \end{aligned} \tag{1}$$

with the credibility factors

$$\begin{aligned} z_{ij} &= \frac{w_{ij\Sigma}}{w_{ijk\Sigma} + s^2/a}, & w_{ij\Sigma} &= \sum_{t=1}^{n_{ij}} w_{ijt} \\ z_i &= \frac{z_{i\Sigma}}{z_{i\Sigma} + a/b}, & z_{i\Sigma} &= \sum_{j=1}^{J_i} z_{ij} \end{aligned}$$

and the weighted averages

$$\begin{aligned} X_{ijw} &= \sum_{t=1}^{n_{ij}} \frac{w_{ijt}}{w_{ij\Sigma}} X_{ijt} \\ X_{izw} &= \sum_{j=1}^{J_i} \frac{z_{ij}}{z_{i\Sigma}} X_{ijw}. \end{aligned}$$

The estimator of s^2 is

$$\hat{s}^2 = \frac{1}{\sum_{i=1}^{I} \sum_{j=1}^{J_i} (n_{ij} - 1)} \sum_{i=1}^{I} \sum_{j=1}^{J_i} \sum_{t=1}^{n_{ij}} w_{ijt} (X_{ijt} - X_{ijw})^2. \tag{2}$$

The three types of estimators for parameters a and b are the following. First, let

$$A_i = \sum_{j=1}^{J_i} w_{ij\Sigma}(X_{ijw} - X_{iww})^2 - (J_i - 1)s^2 \qquad c_i = w_{i\Sigma\Sigma} - \sum_{j=1}^{J_i} \frac{w_{ij\Sigma}^2}{w_{i\Sigma\Sigma}}$$

$$B = \sum_{i=1}^{I} z_{i\Sigma}(X_{izw} - \bar{X}_{zzw})^2 - (I-1)a \qquad d = z_{\Sigma\Sigma} - \sum_{i=1}^{I} \frac{z_{i\Sigma}^2}{z_{\Sigma\Sigma}},$$

with

$$\bar{X}_{zzw} = \sum_{i=1}^{I} \frac{z_{i\Sigma}}{z_{\Sigma\Sigma}} X_{izw}. \tag{3}$$

(Hence, $\mathrm{E}[A_i] = c_i a$ and $\mathrm{E}[B] = db$.) Then, the Bühlmann–Gisler estimators are

$$\hat{a} = \frac{1}{I}\sum_{i=1}^{I} \max\left(\frac{A_i}{c_i}, 0\right) \tag{4}$$

$$\hat{b} = \max\left(\frac{B}{d}, 0\right), \tag{5}$$

the Ohlsson estimators are

$$\hat{a}' = \frac{\sum_{i=1}^{I} A_i}{\sum_{i=1}^{I} c_i} \tag{6}$$

$$\hat{b}' = \frac{B}{d} \tag{7}$$

and the iterative (pseudo-)estimators are

$$\tilde{a} = \frac{1}{\sum_{i=1}^{I}(J_i - 1)} \sum_{i=1}^{I}\sum_{j=1}^{J_i} z_{ij}(X_{ijw} - X_{izw})^2 \tag{8}$$

$$\tilde{b} = \frac{1}{I-1}\sum_{i=1}^{I} z_i (X_{izw} - X_{zzw})^2, \tag{9}$$

where

$$X_{zzw} = \sum_{i=1}^{I} \frac{z_i}{z_\Sigma} X_{izw}. \tag{10}$$

Note the difference between the two weighted averages (3) and (10). See Belhadj et al. (2009) for further discussion on this topic.

Finally, the estimator of the collective mean m is $\hat{m} = X_{zzw}$.

The credibility modeling function `cm` assumes that data is available in the format most practical applications would use, namely a rectangular array (matrix or data frame) with entity observations in the rows and with one or more classification index columns (numeric or character). One will recognize the output format of `simul` and its summary methods.

Then, function `cm` works much the same as `lm`. It takes in argument: a formula of the form `~ terms` describing the hierarchical interactions in a data set; the data set containing the variables referenced in the formula; the names of the columns where the ratios and the weights are to be found in the data set. The latter should contain at least two nodes in each level and more than one period of experience for at least one entity. Missing values are represented by `NA`s. There can be entities with no experience (complete lines of `NA`s).

In order to give an easily reproducible example, we group states 1 and 3 of the Hachemeister data set into one cohort and states 2, 4 and 5 into another. This shows that data does not have to be sorted by level. The fitted model using the iterative estimators is:

```
\begin{Sinput}
> X <- cbind(cohort = c(1, 2, 1, 2, 2), hachemeister)
> fit <- cm(~cohort + cohort:state, data = X,
+           ratios = ratio.1:ratio.12,
+           weights = weight.1:weight.12,
+           method = "iterative")
> fit
\end{Sinput}
\begin{Soutput}
Call:
cm(formula = ~cohort + cohort:state, data = X, ratios = ratio.1:ratio.12,
    weights = weight.1:weight.12, method = "iterative")

Structure Parameters Estimators

  Collective premium: 1746

  Between cohort variance: 88981
  Within cohort/Between state variance: 10952
  Within state variance: 139120026
\end{Soutput}
```

The function returns a fitted model object of class `"cm"` containing the estimators of the structure parameters. To compute the credibility premiums, one calls a method of `predict` for this class:

```
\begin{Sinput}
> predict(fit)
\end{Sinput}
\begin{Soutput}
$cohort
[1] 1949 1543

$state
[1] 2048 1524 1875 1497 1585
\end{Soutput}
```

One can also obtain a nicely formatted view of the most important results with a call to `summary`:

```
\begin{Sinput}
> summary(fit)
\end{Sinput}
\begin{Soutput}
Call:
cm(formula = ~cohort + cohort:state, data = X, ratios = ratio.1:ratio.12,
    weights = weight.1:weight.12, method = "iterative")

Structure Parameters Estimators

  Collective premium: 1746

  Between cohort variance: 88981
  Within cohort/Between state variance: 10952
  Within state variance: 139120026

Detailed premiums

  Level: cohort
    cohort Indiv. mean Weight Cred. factor Cred. premium
    1      1967        1.407  0.9196       1949
    2      1528        1.596  0.9284       1543

  Level: state
    cohort state Indiv. mean Weight Cred. factor
    1      1     2061        100155 0.8874
    2      2     1511         19895 0.6103
    1      3     1806         13735 0.5195
    2      4     1353          4152 0.2463
    2      5     1600         36110 0.7398
    Cred. premium
    2048
    1524
    1875
    1497
    1585
\end{Soutput}
```

The methods of `predict` and `summary` can both report for a subset of the levels by means of an argument `levels`. For example:

```
\begin{Sinput}
> summary(fit, levels = "cohort")
\end{Sinput}
\begin{Soutput}
Call:
cm(formula = ~cohort + cohort:state, data = X, ratios = ratio.1:ratio.12,
    weights = weight.1:weight.12, method = "iterative")

Structure Parameters Estimators

  Collective premium: 1746

  Between cohort variance: 88981
  Within cohort variance: 10952

Detailed premiums

  Level: cohort
    cohort Indiv. mean Weight Cred. factor Cred. premium
    1      1967        1.407  0.9196       1949
    2      1528        1.596  0.9284       1543
\end{Soutput}
\begin{Sinput}
> predict(fit, levels = "cohort")
\end{Sinput}
\begin{Soutput}
$cohort
[1] 1949 1543
\end{Soutput}
```

The results above differ from those of Goovaerts and Hoogstad (1987) for the same example because the formulas for the credibility premiums are different.

4 Bühlmann and Bühlmann–Straub models

As mentioned above, the Bühlmann and Bühlmann–Straub models are simply one-level hierarchical models. In this case, the Bühlmann–Gisler and Ohlsson estimators of the between variance parameters are both identical to the usual Bühlmann and Straub (1970) estimator

$$\hat{a} = \frac{w_{\Sigma\Sigma}}{w_{\Sigma\Sigma}^2 - \sum_{i=1}^{I} w_{i\Sigma}^2} \left(\sum_{i=1}^{I} w_{i\Sigma}(X_{iw} - X_{ww})^2 - (I-1)\hat{s}^2 \right), \tag{11}$$

and the iterative estimator

$$\tilde{a} = \frac{1}{I-1} \sum_{i=1}^{I} z_i (X_{iw} - X_{zw})^2 \tag{12}$$

is better known as the Bichsel–Straub estimator.

To fit the Bühlmann model using `cm`, one simply does not specify any weights:

```
\begin{Sinput}
> cm(~state, hachemeister, ratios = ratio.1:ratio.12)
\end{Sinput}
\begin{Soutput}
Call:
cm(formula = ~state, data = hachemeister, ratios = ratio.1:ratio.12)

Structure Parameters Estimators

  Collective premium: 1671

  Between state variance: 72310
  Within state variance: 46040
\end{Soutput}
```

In comparison, the results for the Bühlmann–Straub model using the Bichsel–Straub estimator are:

```
\begin{Sinput}
> cm(~state, hachemeister, ratios = ratio.1:ratio.12,
+    weights = weight.1:weight.12)
\end{Sinput}
\begin{Soutput}
Call:
cm(formula = ~state, data = hachemeister, ratios = ratio.1:ratio.12,
    weights = weight.1:weight.12)

Structure Parameters Estimators

  Collective premium: 1684

  Between state variance: 89639
  Within state variance: 139120026
\end{Soutput}
```

5 Regression model of Hachemeister

The regression model of Hachemeister (1975) is a generalization of the Bühlmann–Straub model. If data shows a systematic trend, the latter model will typically under- or over-estimate the true premium of an entity. The idea of Hachemeister was to fit to the data a regression model where the parameters are a credibility weighted average of an entity's regression parameters and the group's parameters.

In order to use `cm` to fit a credibility regression model to a data set, one simply has to supply as additional arguments `regformula` and `regdata`. The first one is a formula of the form `~ terms` describing the regression model and the second is a data frame of regressors. That is, arguments `regformula` and `regdata` are in every respect equivalent to arguments `formula` and `data` of `lm`, with the minor difference that `regformula` does not need to have a left hand side (and is ignored if present). For example, fitting the model

$$X_{it} = \beta_0 + \beta_1 t + \varepsilon_t, \quad t = 1, \dots, 12$$

to the original data set of Hachemeister (1975) is done with

```
\begin{Sinput}
> fit <- cm(~state, hachemeister, regformula = ~ time,
+           regdata = data.frame(time = 1:12),
+           ratios = ratio.1:ratio.12,
+           weights = weight.1:weight.12)
> fit
\end{Sinput}
\begin{Soutput}
Call:
cm(formula = ~state, data = hachemeister, ratios = ratio.1:ratio.12,
    weights = weight.1:weight.12, regformula = ~time, regdata = data.frame(time = 1:12))

Structure Parameters Estimators

  Collective premium: 1469 32.05

  Between state variance:  24154 2700.0
                            2700  301.8
  Within state variance: 49870187
\end{Soutput}
```

Computing the credibility premiums requires to give the "future" values of the regressors as in `predict.lm`:

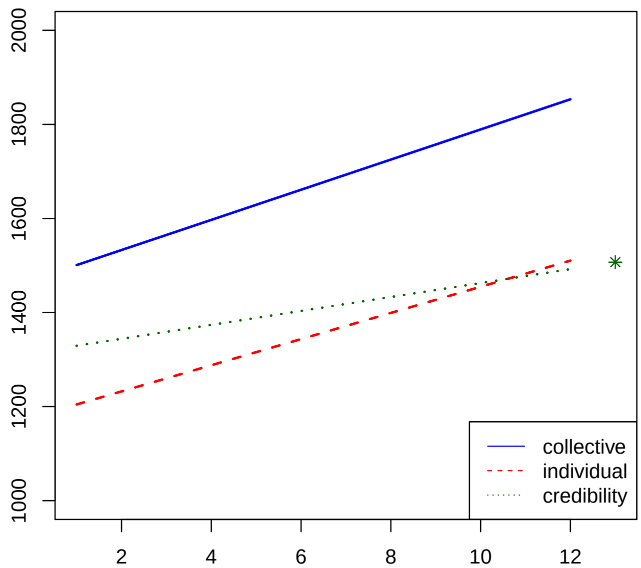

Figure 1: Collective, individual and credibility regression lines for State 4 of the Hachemeister data set. The point indicates the credibility premium.

```
\begin{Sinput}
> predict(fit, newdata = data.frame(time = 13))
\end{Sinput}
\begin{Soutput}
[1] 2437 1651 2073 1507 1759
\end{Soutput}
```

It is well known that the basic regression model has a major drawback: there is no guarantee that the credibility regression line will lie between the collective and individual ones. This may lead to grossly inadequate premiums, as Figure 1 shows.

The solution proposed by Bühlmann and Gisler (1997) is simply to position the intercept at the barycenter of time instead of at time origin (see also Bühlmann and Gisler, 2005, Section 8.4). In mathematical terms, this essentially amounts to using an orthogonal design matrix. By setting the argument `adj.intercept` to `TRUE` in the call, `cm` will automatically fit the credibility regression model with the intercept at the barycenter of time. The resulting regression coefficients have little meaning, but the predictions are sensible:

```
\begin{Sinput}
> fit2 <- cm(~state, hachemeister, regformula = ~ time,
+            regdata = data.frame(time = 1:12),
+            adj.intercept = TRUE,
+            ratios = ratio.1:ratio.12,
+            weights = weight.1:weight.12)
> summary(fit2, newdata = data.frame(time = 13))
\end{Sinput}
\begin{Soutput}
Call:
cm(formula = ~state, data = hachemeister, ratios = ratio.1:ratio.12,
    weights = weight.1:weight.12, regformula = ~time, regdata = data.frame(time = 1:12),
    adj.intercept = TRUE)

Structure Parameters Estimators

  Collective premium: -1675 117.1

  Between state variance:  93783    0
                               0 8046
  Within state variance: 49870187

Detailed premiums

  Level: state
    state Indiv. coef. Credibility matrix Adj. coef.
    1     -2062.46     0.9947 0.0000      -2060.41
            216.97     0.0000 0.9413        211.10
    2     -1509.28     0.9740 0.0000      -1513.59
             59.60     0.0000 0.7630         73.23
    3     -1813.41     0.9627 0.0000      -1808.25
            150.60     0.0000 0.6885        140.16
    4     -1356.75     0.8865 0.0000      -1392.88
             96.70     0.0000 0.4080        108.77
    5     -1598.79     0.9855 0.0000      -1599.89
             41.29     0.0000 0.8559         52.22
    Cred. premium
    2457

    1651

    2071

    1597

    1698
\end{Soutput}
```

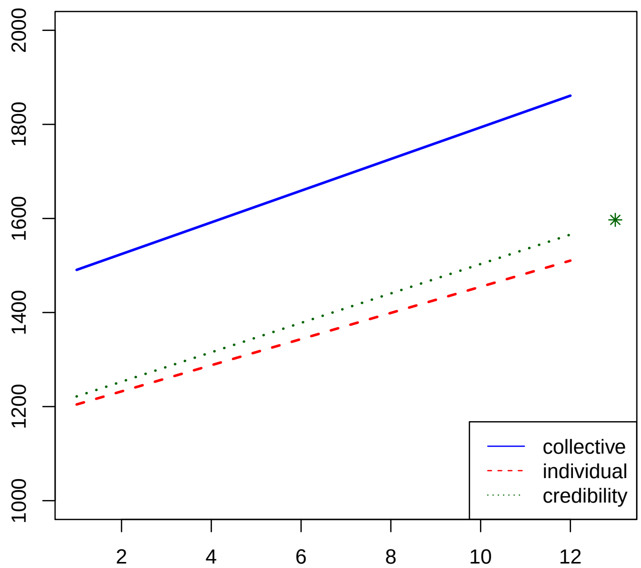

Figure 2: Collective, individual and credibility regression lines for State 4 of the Hachemeister data set when the intercept is positioned at the barycenter of time. The point indicates the credibility premium.

Figure 2 shows the beneficient effect of the intercept adjustment on the premium of State 4.

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