Credibility theory features of **actuar**

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1 Introduction

Credibility models are actuarial tools to distribute premiums fairly among a heterogeneous group of policyholders (henceforth called *entities*). More generally, they can be seen as prediction methods applicable in any setting where repeated measures are made for subjects with different risk levels.

The credibility theory features of **actuar** consist of matrix `hachemeister` containing the famous data set of Hachemeister (1975) and function `cm` to fit hierarchical (including Bühlmann, Bühlmann-Straub) and regression credibility models. Furthermore, function `simul` can simulate portfolios of data satisfying the assumptions of the aforementioned credibility models; see the `"simulation"` vignette for details.

2 Hachemeister data set

The data set of Hachemeister (1975) consists of private passenger bodily injury insurance average claim amounts, and the corresponding number of claims, for five U.S. states over 12 quarters between July 1970 and June 1973. The data set is included in the package in the form of a matrix with 5 rows and 25 columns. The first column contains a state index, columns 2–13 contain the claim averages and columns 14–25 contain the claim numbers:

```
> data(hachemeister)
> hachemeister

     state ratio.1 ratio.2 ratio.3 ratio.4 ratio.5 ratio.6 ratio.7
[1,]     1    1738    1642    1794    2051    2079    2234    2032
[2,]     2    1364    1408    1597    1444    1342    1675    1470
[3,]     3    1759    1685    1479    1763    1674    2103    1502
[4,]     4    1223    1146    1010    1257    1426    1532    1953
[5,]     5    1456    1499    1609    1741    1482    1572    1606
     ratio.8 ratio.9 ratio.10 ratio.11 ratio.12 weight.1 weight.2
[1,]    2035    2115     2262     2267     2517     7861     9251
[2,]    1448    1464     1831     1612     1471     1622     1742
[3,]    1622    1828     2155     2233     2059     1147     1357
[4,]    1123    1343     1243     1762     1306      407      396
[5,]    1735    1607     1573     1613     1690     2902     3172
     weight.3 weight.4 weight.5 weight.6 weight.7 weight.8 weight.9
[1,]     8706     8575     7917     8263     9456     8003     7365
[2,]     1523     1515     1622     1602     1964     1515     1527
[3,]     1329     1204      998     1077     1277     1218      896
[4,]      348      341      315      328      352      331      287
[5,]     3046     3068     2693     2910     3275     2697     2663
     weight.10 weight.11 weight.12
[1,]      7832      7849      9077
[2,]      1748      1654      1861
[3,]      1003      1108      1121
[4,]       384       321       342
[5,]      3017      3242      3425
```

3 Hierarchical credibility model

The linear model fitting function of R is named lm. Since credibility models are very close in many respects to linear models, and since the credibility model fitting function of **actuar** borrows much of its interface from lm, we named the credibility function cm.

Function cm acts as a unified interface for all credibility models supported by the package. Currently, these are the unidimensional models of Bühlmann (1969) and Bühlmann and Straub (1970), the hierarchical model of Jewell (1975) (of which the first two are special cases) and the regression model of Hachemeister (1975), optionally with the intercept at the barycenter of time (Bühlmann and Gisler, 2005, Section 8.4). The modular design of cm makes it easy to add new models if desired.

This subsection concentrates on usage of cm for hierarchical models.

There are some variations in the formulas of the hierarchical model in the literature. We compute the credibility premiums as given in Bühlmann and Jewell (1987) or Bühlmann and Gisler (2005). We support three types of estimators of the between variance structure parameters: the unbiased estimators of Bühlmann and Gisler (2005) (the default), the slightly different version of

Ohlsson (2005) and the iterative pseudo-estimators as found in Goovaerts and Hoogstad (1987) or Goulet (1998).

Consider an insurance portfolio where contracts are classified into cohorts. In our terminology, this is a two-level hierarchical classification structure. The observations are claim amounts S_{ijt}, where index $i = 1, \dots, I$ identifies the cohort, index $j = 1, \dots, J_i$ identifies the contract within the cohort and index $t = 1, \dots, n_{ij}$ identifies the period (usually a year). To each data point corresponds a weight — or volume – w_{ijt}. Then, the best linear prediction for the next period outcome of a contract based on ratios $X_{ijt} = S_{ijt}/w_{ijt}$ is

$$\begin{aligned} \hat{\pi}_{ij} &= z_{ij} X_{ijw} + (1 - z_{ij}) \hat{\pi}_i \\ \hat{\pi}_i &= z_i X_{izw} + (1 - z_i) m \end{aligned} \tag{1}$$

with the credibility factors

$$z_{ij} = \frac{w_{ij\Sigma}}{w_{ijk\Sigma} + s^2/a}, \qquad w_{ij\Sigma} = \sum_{t=1}^{n_{ij}} w_{ijt}$$

$$z_i = \frac{z_{i\Sigma}}{z_{i\Sigma} + a/b}, \qquad z_{i\Sigma} = \sum_{j=1}^{J_i} z_{ij}$$

and the weighted averages

$$X_{ijw} = \sum_{t=1}^{n_{ij}} \frac{w_{ijt}}{w_{ij\Sigma}} X_{ijt}$$

$$X_{izw} = \sum_{j=1}^{J_i} \frac{z_{ij}}{z_{i\Sigma}} X_{ijw}.$$

The estimator of s^2 is

$$\hat{s}^2 = \frac{1}{\sum_{i=1}^{I} \sum_{j=1}^{J_i} (n_{ij} - 1)} \sum_{i=1}^{I} \sum_{j=1}^{J_i} \sum_{t=1}^{n_{ij}} w_{ijt} (X_{ijt} - X_{ijw})^2. \tag{2}$$

The three types of estimators for parameters a and b are the following. First, let

$$A_i = \sum_{j=1}^{J_i} w_{ij\Sigma} (X_{ijw} - X_{iww})^2 - (J_i - 1) s^2 \qquad c_i = w_{i\Sigma\Sigma} - \sum_{j=1}^{J_i} \frac{w_{ij\Sigma}^2}{w_{i\Sigma\Sigma}}$$

$$B = \sum_{i=1}^{I} z_{i\Sigma} (X_{izw} - \bar{X}_{zzw})^2 - (I - 1) a \qquad d = z_{\Sigma\Sigma} - \sum_{i=1}^{I} \frac{z_{i\Sigma}^2}{z_{\Sigma\Sigma}},$$

with

$$\bar{X}_{zzw} = \sum_{i=1}^{I} \frac{z_{i\Sigma}}{z_{\Sigma\Sigma}} X_{izw}. \tag{3}$$

(Hence, $\mathrm{E}[A_i] = c_i a$ and $\mathrm{E}[B] = db$.) Then, the Bühlmann–Gisler estimators are

$$\hat{a} = \frac{1}{I} \sum_{i=1}^{I} \max\left(\frac{A_i}{c_i}, 0 \right) \tag{4}$$

$$\hat{b} = \max\left(\frac{B}{d}, 0 \right), \tag{5}$$

the Ohlsson estimators are

$$\hat{a}' = \frac{\sum_{i=1}^{I} A_i}{\sum_{i=1}^{I} c_i} \tag{6}$$

$$\hat{b}' = \frac{B}{d} \tag{7}$$

and the iterative (pseudo-)estimators are

$$\tilde{a} = \frac{1}{\sum_{i=1}^{I} (J_i - 1)} \sum_{i=1}^{I} \sum_{j=1}^{J_i} z_{ij} (X_{ijw} - X_{izw})^2 \tag{8}$$

$$\tilde{b} = \frac{1}{I-1} \sum_{i=1}^{I} z_i (X_{izw} - X_{zzw})^2, \tag{9}$$

where

$$X_{zzw} = \sum_{i=1}^{I} \frac{z_i}{z_\Sigma} X_{izw}. \tag{10}$$

Note the difference between the two weighted averages (3) and (10). See Belhadj et al. (2009) for further discussion on this topic.

Finally, the estimator of the collective mean m is $\hat{m} = X_{zzw}$.

The credibility modeling function `cm` assumes that data is available in the format most practical applications would use, namely a rectangular array (matrix or data frame) with entity observations in the rows and with one or more classification index columns (numeric or character). One will recognize the output format of `simul` and its summary methods.

Then, function `cm` works much the same as `lm`. It takes in argument: a formula of the form `~ terms` describing the hierarchical interactions in a data set; the data set containing the variables referenced in the formula; the names of the columns where the ratios and the weights are to be found in the data set. The latter should contain at least two nodes in each level and more than one period of experience for at least one entity. Missing values are represented by `NA`s. There can be entities with no experience (complete lines of `NA`s).

In order to give an easily reproducible example, we group states 1 and 3 of the Hachemeister data set into one cohort and states 2, 4 and 5 into another. This shows that data does not have to be sorted by level. The fitted model using the iterative estimators is:

```
> X <- cbind(cohort = c(1, 2, 1, 2, 2), hachemeister)
> fit <- cm(~cohort + cohort:state, data = X, ratios = ratio.1:ratio.12,
+           weights = weight.1:weight.12, method = "iterative")
> fit

Call:
cm(formula = ~cohort + cohort:state, data = X, ratios = ratio.1:ratio.12,
    weights = weight.1:weight.12, method = "iterative")

Structure Parameters Estimators

  Collective premium: 1746

  Between cohort variance: 88981
  Within cohort/Between state variance: 10952
  Within state variance: 139120026
```

The function returns a fitted model object of class "cm" containing the estimators of the structure parameters. To compute the credibility premiums, one calls a method of predict for this class:

```
> predict(fit)

$cohort
[1] 1949 1543

$state
[1] 2048 1524 1875 1497 1585
```

One can also obtain a nicely formatted view of the most important results with a call to summary:

```
> summary(fit)

Call:
cm(formula = ~cohort + cohort:state, data = X, ratios = ratio.1:ratio.12,
    weights = weight.1:weight.12, method = "iterative")

Structure Parameters Estimators

  Collective premium: 1746

  Between cohort variance: 88981
  Within cohort/Between state variance: 10952
  Within state variance: 139120026

Detailed premiums

  Level: cohort
    cohort Indiv. mean Weight Cred. factor Cred. premium
    1      1967        1.407  0.9196       1949
```

```
    2      1528        1.596  0.9284       1543

  Level: state
    cohort state Indiv. mean Weight Cred. factor Cred. premium
    1      1     2061        100155 0.8874       2048
    2      2     1511         19895 0.6103       1524
    1      3     1806         13735 0.5195       1875
    2      4     1353          4152 0.2463       1497
    2      5     1600         36110 0.7398       1585
```

The methods of `predict` and `summary` can both report for a subset of the levels by means of an argument `levels`. For example:

```
> summary(fit, levels = "cohort")

Call:
cm(formula = ~cohort + cohort:state, data = X, ratios = ratio.1:ratio.12,
    weights = weight.1:weight.12, method = "iterative")

Structure Parameters Estimators

  Collective premium: 1746

  Between cohort variance: 88981
  Within cohort variance: 10952

Detailed premiums

  Level: cohort
    cohort Indiv. mean Weight Cred. factor Cred. premium
    1      1967        1.407  0.9196       1949
    2      1528        1.596  0.9284       1543

> predict(fit, levels = "cohort")

$cohort
[1] 1949 1543
```

The results above differ from those of Goovaerts and Hoogstad (1987) for the same example because the formulas for the credibility premiums are different.

4 Bühlmann and Bühlmann–Straub models

As mentioned above, the Bühlmann and Bühlmann–Straub models are simply one-level hierarchical models. In this case, the Bühlmann–Gisler and Ohlsson estimators of the between variance parameters are both identical to the usual Bühlmann and Straub (1970) estimator

$$\hat{a} = \frac{w_{\Sigma\Sigma}}{w_{\Sigma\Sigma}^2 - \sum_{i=1}^I w_{i\Sigma}^2} \left(\sum_{i=1}^I w_{i\Sigma}(X_{iw} - X_{ww})^2 - (I-1)\hat{s}^2 \right), \tag{11}$$

and the iterative estimator

$$\tilde{a} = \frac{1}{I-1} \sum_{i=1}^{I} z_i (X_{iw} - X_{zw})^2 \qquad (12)$$

is better known as the Bichsel–Straub estimator.

To fit the Bühlmann model using cm, one simply does not specify any weights:

```
> cm(~state, hachemeister, ratios = ratio.1:ratio.12)

Call:
cm(formula = ~state, data = hachemeister, ratios = ratio.1:ratio.12)

Structure Parameters Estimators

  Collective premium: 1671

  Between state variance: 72310
  Within state variance: 46040
```

In comparison, the results for the Bühlmann–Straub model using the Bichsel–Straub estimator are:

```
> cm(~state, hachemeister, ratios = ratio.1:ratio.12,
+    weights = weight.1:weight.12)

Call:
cm(formula = ~state, data = hachemeister, ratios = ratio.1:ratio.12,
    weights = weight.1:weight.12)

Structure Parameters Estimators

  Collective premium: 1684

  Between state variance: 89639
  Within state variance: 139120026
```

5 Regression model of Hachemeister

The regression model of Hachemeister (1975) is a generalization of the Bühlmann–Straub model. If data shows a systematic trend, the latter model will typically under- or over-estimate the true premium of an entity. The idea of Hachemeister was to fit to the data a regression model where the parameters are a credibility weighted average of an entity's regression parameters and the group's parameters.

In order to use cm to fit a credibility regression model to a data set, one simply has to supply as additional arguments regformula and regdata. The first one is a formula of the form `~ terms` describing the regression model and the second is a data frame of regressors. That is, arguments `regformula` and `regdata` are in every respect equivalent to arguments `formula` and `data` of `lm`, with the minor difference that `regformula` does not need to have a left hand side (and is ignored if present). For example, fitting the model

$$X_{it} = \beta_0 + \beta_1 t + \varepsilon_t, \quad t = 1, \dots, 12$$

to the original data set of Hachemeister (1975) is done with

```
> fit <- cm(~state, hachemeister,
+           regformula = ~ time, regdata = data.frame(time = 1:12),
+           ratios = ratio.1:ratio.12, weights = weight.1:weight.12)
> fit

Call:
cm(formula = ~state, data = hachemeister, ratios = ratio.1:ratio.12,
    weights = weight.1:weight.12, regformula = ~time, regdata = data.frame(time = 1:12))

Structure Parameters Estimators

  Collective premium: 1469 32.05

  Between state variance:  24154 2700.0
                            2700  301.8
  Within state variance: 49870187
```

Computing the credibility premiums requires to give the "future" values of the regressors as in `predict.lm`:

```
> predict(fit, newdata = data.frame(time = 13))

[1] 2437 1651 2073 1507 1759
```

It is well known that the basic regression model has a major drawback: there is no guarantee that the credibility regression line will lie between the collective and individual ones. This may lead to grossly inadequate premiums, as Figure 1 shows.

The solution proposed by Bühlmann and Gisler (1997) is simply to position the intercept at the barycenter of time instead of at time origin (see also Bühlmann and Gisler, 2005, Section 8.4). In mathematical terms, this essentially amounts to using an orthogonal design matrix. By setting the argument `adj.intercept` to `TRUE` in the call, `cm` will automatically fit the credibility regression model with the intercept at the barycenter of time. The resulting regression coefficients have little meaning, but the predictions are sensible:

```
> fit2 <- cm(~state, hachemeister,
+            regformula = ~ time, regdata = data.frame(time = 1:12),
+            adj.intercept = TRUE,
+            ratios = ratio.1:ratio.12, weights = weight.1:weight.12)
> summary(fit2, newdata = data.frame(time = 13))
```

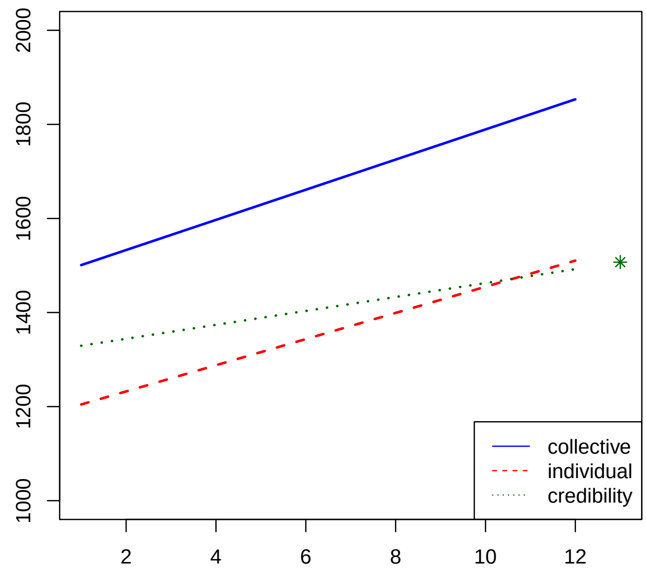

Figure 1: Collective, individual and credibility regression lines for State 4 of the Hachemeister data set. The point indicates the credibility premium.

```
Call:
cm(formula = ~state, data = hachemeister, ratios = ratio.1:ratio.12,
    weights = weight.1:weight.12, regformula = ~time, regdata = data.frame(time = 1:12),
    adj.intercept = TRUE)

Structure Parameters Estimators

  Collective premium: -1675 117.1

  Between state variance:  93783    0
                               0 8046
  Within state variance: 49870187

Detailed premiums

  Level: state
    state Indiv. coef. Credibility matrix Adj. coef. Cred. premium
    1     -2062.46     0.9947 0.0000      -2060.41   2457
            216.97     0.0000 0.9413        211.10
```

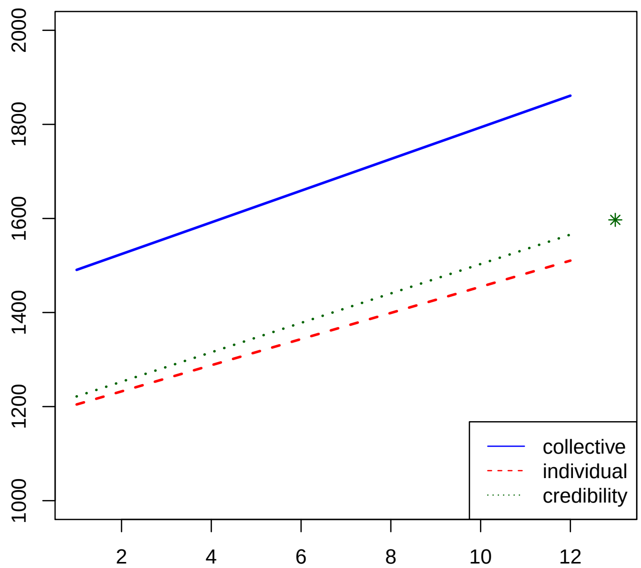

Figure 2: Collective, individual and credibility regression lines for State 4 of the Hachemeister data set when the intercept is positioned at the barycenter of time. The point indicates the credibility premium.

```
2      -1509.28       0.9740 0.0000       -1513.59    1651
          59.60       0.0000 0.7630          73.23
3      -1813.41       0.9627 0.0000       -1808.25    2071
         150.60       0.0000 0.6885         140.16
4      -1356.75       0.8865 0.0000       -1392.88    1597
          96.70       0.0000 0.4080         108.77
5      -1598.79       0.9855 0.0000       -1599.89    1698
          41.29       0.0000 0.8559          52.22
```

Figure 2 shows the beneficient effect of the intercept adjustment on the premium of State 4.

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