Stocking-Lord

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Stocking-Lord is an iterative procedure that minimizes the distance between two test characteristic curves. This method has been implemented in `R` via the function `SL` in the **MiscPsycho** package as described in this section.

Treating θ as a continuous variable, it is possible to introduce a population distribution $f(\theta)$ and use the law of total probability to integrate θ out of the function and then perform the minimization as follows:

$$SL = \int \left[\sum_{i=1}^{I} p(\theta; a_{ia}, b_{ia}, c_{ia}) - \sum_{i=1}^{I} p\left(\theta; \frac{a_{ib}}{A}, Ab_{ib} + B, c_{ib}\right) \right]^2 f(\theta) d\theta \tag{1}$$

where i indexes item, $i = (1, \ldots, I)$, a and b index test forms, $f(\theta) \sim N(\mu, \sigma^2)$ is the mean and variance of the population distribution, and A and B are the linking constants.

The integral in the function SL cannot be evaluated easily. For that reason it is approximated using Gauss-Hermite quadrature as follows:

$$SL \approx \sum_{q=1}^{Q} \left[\sum_{i=1}^{I} p(\theta_q; a_{ia}, b_{ia}, c_{ia}) - \sum_{i=1}^{I} p\left(\theta_q; \frac{a_{ib}}{A}, Ab_{ib} + B, c_{ib}\right) \right]^2 w_q \tag{2}$$

where q indexes quadrature point, $q = (1, \ldots, Q)$ and w_q is the weight at quadrature point q.

Minimization of the function SL with respect to A and B is implemented using Newton-Raphson iterations as:

$$\boldsymbol{L_{t+1}} = \boldsymbol{L_t} - \boldsymbol{H_t}^{-1} \boldsymbol{g_t}^T \tag{3}$$

where the subscript denotes iteration t and:

$$\boldsymbol{L} = [\hat{A}, \hat{B}] \tag{4}$$

$$\boldsymbol{g} = \left[\frac{\partial SL}{\partial A}, \frac{\partial SL}{\partial B}\right] \tag{5}$$

$$\boldsymbol{H} = \left[\begin{array}{cc} \frac{\partial^2 SL}{\partial A} & \frac{\partial^2 SL}{\partial B,\partial A} \\ \frac{\partial^2 SL}{\partial A,\partial B} & \frac{\partial^2 SL}{\partial B} \end{array}\right] \tag{6}$$

The required derivatives for the 3-parameter logistic model are:

$$\begin{aligned}
\frac{\partial SL}{\partial A} &= 2\left[\sum_{i=1}^{I} c_{ia} + \frac{1-c_{ia}}{1+\exp[-Da_{ia}(-b_{ia}-\theta)]} - \sum_{i=1}^{I} c_{ib} + \frac{1-c_{ib}}{1+\exp\left[\frac{-Da_{ib}(-B-Ab_{ib}+\theta)}{A}\right]}\right] \\
&\times \sum_{i=1}^{I}\left[\frac{(1-c_{ib})\exp\left[\frac{-Da_{ib}(-B-Ab_{ib}+\theta)}{A}\right]\left[\frac{Da_{bi}b_{bi}}{A} + \frac{Da_{ib}(-B-Ab_{ib}+\theta)}{A^2}\right]}{\left(1+\exp\left[\frac{-Da_{ib}(-B-Ab_{ib}+\theta)}{A}\right]\right)^2}\right]
\end{aligned}$$

$$\begin{aligned}
\frac{\partial SL}{\partial B} &= \left[\sum_{i=1}^{I} c_{ia} + \frac{1-c_{ia}}{1+\exp[-Da_{ia}(-b_{ia}-\theta)]} - \sum_{i=1}^{I} c_{ib} + \frac{1-c_{ib}}{1+\exp\left[\frac{-Da_{ib}(-B-Ab_{ib}+\theta)}{A}\right]}\right] \\
&\times 2\sum_{i=1}^{I}\left[\frac{Da_{bi}(1-c_{bi})\exp\left[\frac{-Da_{ib}(-B-Ab_{ib}+\theta)}{A}\right]}{A\left(1+\exp\left[\frac{-Da_{ib}(-B-Ab_{ib}+\theta)}{A}\right]\right)^2}\right]
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 SL}{\partial A} &= \sum_{i=1}^{I}\left[\frac{(1-c_{ib})\exp\left[\frac{-Da_{ib}(-B-Ab_{ib}+\theta)}{A}\right]\left[\frac{Da_{bi}b_{bi}}{A} + \frac{Da_{ib}(-B-Ab_{ib}+\theta)}{A^2}\right]}{\left(1+\exp\left[\frac{-Da_{ib}(-B-Ab_{ib}+\theta)}{A}\right]\right)^2}\right]^2 \\
&+2\left[\sum_{i=1}^{I} c_{ia} + \frac{1-c_{ia}}{1+\exp[-Da_{ia}(-b_{ia}-\theta)]} - \sum_{i=1}^{I} c_{ib} + \frac{1-c_{ib}}{1+\exp\left[\frac{-Da_{ib}(-B-Ab_{ib}+\theta)}{A}\right]}\right] \\
&\times \sum_{i=1}^{I}\left[\frac{(1-c_{ib})\exp\left[\frac{-Da_{ib}(-B-Ab_{ib}+\theta)}{A}\right]\left[\frac{-2Da_{bi}b_{bi}}{A^2} - \frac{2Da_{ib}(-B-Ab_{ib}+\theta)}{A^3}\right]}{\left(1+\exp\left[\frac{-Da_{ib}(-B-Ab_{ib}+\theta)}{A}\right]\right)^3}\right] \\
&+\sum_{i=1}^{I}\left[\frac{(1-c_{ib})\exp\left[\frac{-Da_{ib}(-B-Ab_{ib}+\theta)}{A}\right]\left[\frac{Da_{bi}b_{bi}}{A} + \frac{Da_{ib}(-B-Ab_{ib}+\theta)}{A^2}\right]^2}{\left(1+\exp\left[\frac{-Da_{ib}(-B-Ab_{ib}+\theta)}{A}\right]\right)^3}\right] \\
&-\sum_{i=1}^{I}\left[\frac{2(1-c_{ib})\exp\left[\frac{-2Da_{ib}(-B-Ab_{ib}+\theta)}{A}\right]\left[\frac{Da_{bi}b_{bi}}{A} + \frac{Da_{ib}(-B-Ab_{ib}+\theta)}{A^2}\right]^2}{\left(1+\exp\left[\frac{-Da_{ib}(-B-Ab_{ib}+\theta)}{A}\right]\right)^3}\right]
\end{aligned}$$

$$
\begin{aligned}
\frac{\partial^2 SL}{\partial B} &= 2\sum_{i=1}^{I}\left[\frac{Da_{bi}(1-c_{bi})\exp\left[\frac{-Da_{ib}(-B-Ab_{ib}+\theta)}{A}\right]}{A\left(1+\exp\left[\frac{-Da_{ib}(-B-Ab_{ib}+\theta)}{A}\right]^2\right)}\right]^2 \\
&\quad +2\left(\sum_{i=1}^{I}\left[\frac{D^2a_{bi}^2(1-c_{bi})\exp\left[\frac{-2Da_{ib}(-B-Ab_{ib}+\theta)}{A}\right]}{A^2\left(1+\exp\left[\frac{-Da_{ib}(-B-Ab_{ib}+\theta)}{A}\right]\right)^2}\right]\right. \\
&\quad \left.+\sum_{i=1}^{I}\left[\frac{2D^2a_{bi}^2(1-c_{bi})\exp\left[\frac{-2Da_{ib}(-B-Ab_{ib}+\theta)}{A}\right]}{A^2\left(1+\exp\left[\frac{-Da_{ib}(-B-Ab_{ib}+\theta)}{A}\right]\right)^3}\right]\right) \\
&\quad \times\left[\sum_{i=1}^{I}c_{ia}+\frac{1-c_{ia}}{1+\exp[-Da_{ia}(-b_{ia}-\theta)]}-\sum_{i=1}^{I}c_{ib}+\frac{1-c_{ib}}{1+\exp\left[\frac{-Da_{ib}(-B-Ab_{ib}+\theta)}{A}\right]}\right]
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 SL}{\partial A,\partial B} &= 2\sum_{i=1}^{I}\left[\frac{Da_{bi}(1-c_{bi})\exp\left[\frac{-Da_{ib}(-B-Ab_{ib}+\theta)}{A}\right]}{A\left(1+\exp\left[\frac{-Da_{ib}(-B-Ab_{ib}+\theta)}{A}\right]\right)^2}\right] \\
&\quad \times\sum_{i=1}^{I}\left[\frac{(1-c_{ib})\exp\left[\frac{-Da_{ib}(-B-Ab_{ib}+\theta)}{A}\right]\left[\frac{Da_{bi}b_{bi}}{A}+\frac{Da_{ib}(-B-Ab_{ib}+\theta)}{A^2}\right]}{\left(1+\exp\left[\frac{-Da_{ib}(-B-Ab_{ib}+\theta)}{A}\right]\right)^2}\right] \\
&\quad +2\left[\sum_{i=1}^{I}c_{ia}+\frac{1-c_{ia}}{1+\exp[-Da_{ia}(-b_{ia}-\theta)]}-\sum_{i=1}^{I}c_{ib}+\frac{1-c_{ib}}{1+\exp\left[\frac{-Da_{ib}(-B-Ab_{ib}+\theta)}{A}\right]}\right] \\
&\quad \times\sum_{i=1}^{I}\left[\frac{Da_{bi}(1-c_{ib})\exp\left[\left(\frac{-Da_{ib}(-B-Ab_{ib}+\theta)}{A}\right)\right]\left[\frac{Da_{bi}b_{bi}}{A}+\frac{Da_{ib}(-B-Ab_{ib}+\theta)}{A^2}\right]}{A\left(1+\exp\left[\frac{-Da_{ib}(-B-Ab_{ib}+\theta)}{A}\right]\right)^2}\right] \\
&\quad -\sum_{i=1}^{I}\left[\frac{Da_{bi}(1-c_{bi})\exp\left[\frac{-Da_{ib}(-B-Ab_{ib}+\theta)}{A}\right]}{A^2\left(1+\exp\left[\frac{-Da_{ib}(-B-Ab_{ib}+\theta)}{A}\right]\right)^2}\right] \\
&\quad +\sum_{i=1}^{I}\left[\frac{2Da_{bi}(1-c_{ib})\exp\left[\frac{-2Da_{ib}(-B-Ab_{ib}+\theta)}{A}\right]\left[\frac{Da_{bi}b_{bi}}{A}+\frac{Da_{ib}(-B-Ab_{ib}+\theta)}{A^2}\right]}{A\left(1+\exp\left[\frac{-Da_{ib}(-B-Ab_{ib}+\theta)}{A}\right]\right)^3}\right]
\end{aligned}
$$

Initial starting values for the linking constants A and B are taken from the mean/sigma transformation

$$A = \frac{\sigma(\hat{b}_b)}{\sigma(\hat{b}_a)} \tag{7}$$

$$B = \mu(\hat{b}_b) - A * \mu(\hat{b}_a) \tag{8}$$