Claims reserving with R: ChainLadder-0.1.9 Package Vignette DRAFT

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Abstract

The `ChainLadder` package provides various statistical methods which are typically used for the estimation of outstanding claims reserves in general insurance.

The package has implementations of the Mack-, Munich-, Bootstrap, and multi-variate chain-ladder methods, as well as the loss development factor curve fitting methods of Dave Clark and generalised linear model based reserving models.

This document is still in a draft stage. Any pointers which will help to iron out errors, clarify and make this document more helpful will be much appreciated.

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1 Introduction

1.1 Claims reserving in insurance

Unlike other industries the insurance industry does not sell products as such, but promises. An insurance policy is a promise by the insurer to the policyholder to pay for future claims for an upfront received premium.

As a result insurers don't know the upfront cost of their service, but rely on historical data analysis and judgment to derive a sustainable price for their offering. In General Insurance (or Non-Life Insurance, e.g. motor, property and casualty insurance) most policies run for a period of 12 months. However, the claims payment process can take years or even decades. Therefore often not even the delivery date of their product is known to insurers.

In particular claims arising from casualty insurance can take a long time to settle. Claims can take years to materialise. A complex and costly example are the claims from asbestos liabilities. A research report by a working party of the Institute of Actuaries has estimated that the undiscounted cost of UK mesothelioma-related claims to the UK Insurance Market for the period 2009 to 2050 could be around £10bn [GBB+09]. The cost for asbestos related claims in the US for the worldwide insurance industry was estimate to be around $120bn in 2002 [Mic02].

Thus, it should come to no surprise that the biggest item on the liability side of an insurer's balance sheet is often the provision or reserves for future claims payments. Those reserves can be broken down in case reserves (or out-standings claims), which are losses already reported to the insurance company and incurred but not reported (IBNR) claims.

Over the years several methods have been developed to estimate reserves for insurance claims, see [Sch11], [PR02] for an overview. Changes in regulatory requirements, e.g. Solvency II[1] in Europe, have fostered further research into this topic, with a focus on stochastic and statistical techniques.

2 The `ChainLadder` package

2.1 Motivation

The `ChainLadder` [GMZ14] package provides various statistical methods which are typically used for the estimation of outstanding claims reserves in general insurance. The package started out of presentations given by Markus Gesmann at the Stochastic Reserving Seminar at the Institute of Actuaries in 2007 and 2008, followed by talks at Casualty Actuarial Society (CAS) meetings joined by Dan Murphy in 2008 and Wayne Zhang in 2010.

[1]See `http://ec.europa.eu/internal_market/insurance/solvency/index_en.htm`

Implementing reserving methods in R has several advantages. R provides:

- a rich language for statistical modelling and data manipulations allowing fast prototyping
- a very active user base, which publishes many extension
- many interfaces to data bases and other applications, such as MS Excel
- an established framework for documentation and testing
- workflows with version control systems
- code written in plain text files, allowing effective knowledge transfer
- an effective way to collaborate over the internet
- built in functions to create reproducible research reports[2]
- in combination with other tools such as LaTeX and Sweave easy to set up automated reporting facilities
- access to academic research, which is often first implemented in R

2.2 Brief package overview

This vignette will give the reader a brief overview of the functionality of the `ChainLadder` package. The functions are discussed and explained in more detail in the respective help files and examples, see also [Ges14].

The `ChainLadder` package has implementations of the Mack-, Munich- and Bootstrap chain-ladder methods [Mac93a], [Mac99], [QM04], [EV99]. Since version 0.1.3-3 it provides general multivariate chain ladder models by Wayne Zhang [Zha10]. Version 0.1.4-0 introduced new functions on loss development factor (LDF) fitting methods and Cape Cod by Daniel Murphy following a paper by David Clark [Cla03]. Version 0.1.5-0 has added loss reserving models within the generalized linear model framework following a paper by England and Verrall [EV99] implemented by Wayne Zhang.

The package also offers utility functions to convert quickly tables into triangles, triangles into tables, cumulative into incremental and incremental into cumulative triangles.

A set of demos is shipped with the packages and the list of demos is available via:

```
R> demo(package="ChainLadder")
```

and can be executed via

[2] For an example see the project: Formatted Actuarial Vignettes in R, http://www.favir.net/

```
R> library(ChainLadder)
R> demo("demo name")
```

For more information and examples see the project web site: http://code.google.com/p/chainladder/

2.3 Installation

We can install ChainLadder in the usual way from CRAN, e.g.:

```
R> install.packages('ChainLadder')
```

For more details about installing packages see [Tea12b]. The installation was successful if the command library(ChainLadder) gives you the following message:

```
R> library(ChainLadder)
```

```
ChainLadder version 0.1.9 by:
Markus Gesmann <markus.gesmann@gmail.com>
Wayne Zhang <actuary_zhang@hotmail.com>
Daniel Murphy <danielmarkmurphy@gmail.com>

Type ?ChainLadder to access overall documentation and
vignette('ChainLadder') for the package vignette.

Type demo(ChainLadder) to get an idea of the functionality of this package.
See demo(package='ChainLadder') for a list of more demos.

More information is available on the ChainLadder project web-site:
http://code.google.com/p/chainladder/

To suppress this message use the statement:
suppressPackageStartupMessages(library(ChainLadder))
```

3 Using the ChainLadder package

3.1 Working with triangles

Historical insurance data is often presented in form of a triangle structure, showing the development of claims over time for each exposure (origin) period. An origin period could be the year the policy was sold, or the accident year. Of course the exposure period doesn't have to be yearly, e.g. quarterly or monthly origin periods are also often used. Most reserving methods of the `ChainLadder` package expect triangles as input data sets with development periods along the columns and the origin period in rows. The package comes with several example triangles. The following R command will list them all:

```
R> require(ChainLadder)
R> data(package="ChainLadder")
```

Let's look at one example triangle more closely. The following triangle shows data from the Reinsurance Association of America (RAA):

```
R> ## Sample triangle
R> RAA
```

```
      dev
origin    1     2     3     4     5     6     7     8     9    10
  1981 5012  8269 10907 11805 13539 16181 18009 18608 18662 18834
  1982  106  4285  5396 10666 13782 15599 15496 16169 16704    NA
  1983 3410  8992 13873 16141 18735 22214 22863 23466    NA    NA
  1984 5655 11555 15766 21266 23425 26083 27067    NA    NA    NA
  1985 1092  9565 15836 22169 25955 26180    NA    NA    NA    NA
  1986 1513  6445 11702 12935 15852    NA    NA    NA    NA    NA
  1987  557  4020 10946 12314    NA    NA    NA    NA    NA    NA
  1988 1351  6947 13112    NA    NA    NA    NA    NA    NA    NA
  1989 3133  5395    NA    NA    NA    NA    NA    NA    NA    NA
  1990 2063    NA    NA    NA    NA    NA    NA    NA    NA    NA
```

This matrix shows the known values of loss from each origin year as of the end of the origin year as as of annual evaluations thereafter. For example, the known values of loss originating from the 1988 exposure period are 1351, 6947, and 13112 as of year ends 1988, 1989, and 1990, respectively. The *latest diagonal* – i.e., the vector 18834, 16704, ... 2063 from the upper right to the lower left – shows the most recent evaluation available. The column headings – 1, 2,..., 10 – hold the *ages* (in years) of the observations in the column relative to the beginning of the exposure period. For example, for the 1988 origin year, the age of the 1351 value, evaluated as of 1988-12-31, is three years.

The objective of a reserving exercise is to forecast the future claims development in the bottom right corner of the triangle and potential further developments beyond development age 10. Eventually all claims for a given origin period will be settled, but it is not always obvious to judge how many years or even decades it will take. We speak of long and short tail business depending on the time it takes to pay all claims.

3.1.1 Plotting triangles

The first thing you often want to do is to plot the data to get an overview. For a data set of class `triangle` the `ChainLadder` package provides default plotting methods to give a graphical overview of the data:

```
R> plot(RAA)
```

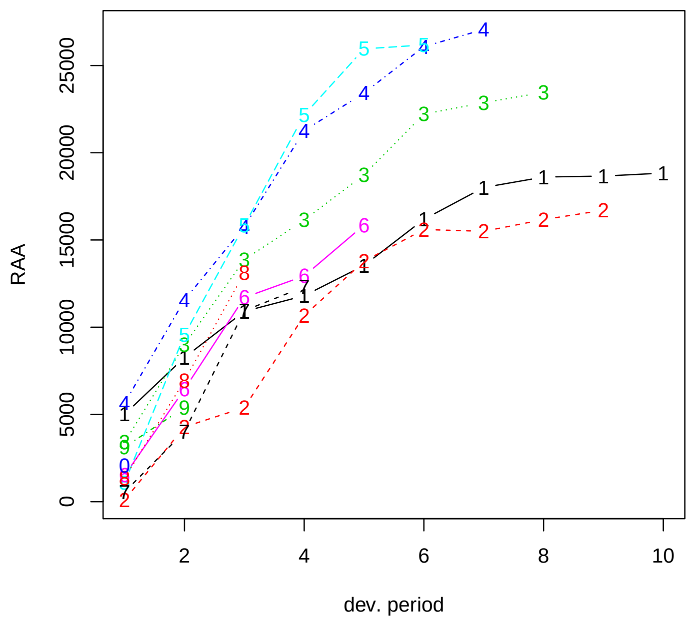


Figure 1: Claims development chart of the `RAA` triangle, with one line per origin period. Output of `plot(RAA)`

Setting the argument `lattice=TRUE` will produce individual plots for each origin period[3], see Figure 2.

```
R> plot(RAA, lattice=TRUE)
```

You will notice from the plots in Figures 1 and 2 that the triangle `RAA` presents claims developments for the origin years 1981 to 1990 in a cumulative form. For more information on the triangle plotting functions see the help pages of `plot.triangle`, e.g. via

[3] `ChainLadder` uses the `lattice` package for plotting the development of the origin years in separate panels.

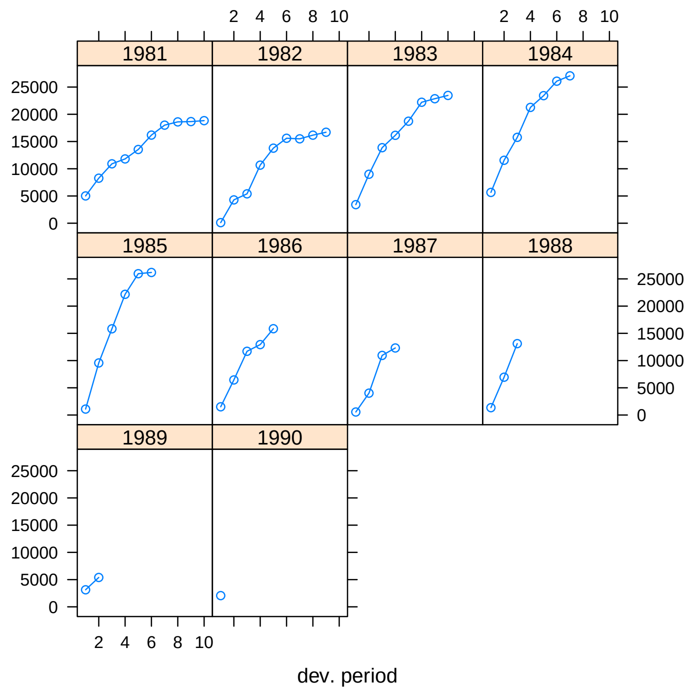


Figure 2: Claims development chart of the RAA triangle, with individual panels for each origin period. Output of `plot(RAA, lattice=TRUE)`

```
R> ?plot.triangle
```

3.1.2 Transforming triangles between cumulative and incremental representation

The `ChainLadder` packages comes with two helper functions, `cum2incr` and `incr2cum` to transform cumulative triangles into incremental triangles and vice versa:

```
R> raa.inc <- cum2incr(RAA)
R> ## Show first origin period and its incremental development
R> raa.inc[1,]

   1    2    3    4    5    6    7    8    9   10
5012 3257 2638  898 1734 2642 1828  599   54  172

R> raa.cum <- incr2cum(raa.inc)
R> ## Show first origin period and its cumulative development
R> raa.cum[1,]
```

```
    1      2      3      4      5      6      7      8      9     10
 5012   8269  10907  11805  13539  16181  18009  18608  18662  18834
```

3.1.3 Importing triangles from external data sources

In most cases you want to analyse your own data, usually stored in data bases. R makes it easy to access data using SQL statements, e.g. via an ODBC connection[4] and the `ChainLadder` packages includes a demo to showcase how data can be imported from a MS Access data base, see:

```
R> demo(DatabaseExamples)
```

For more details see [Tea12a].

In this section we use data stored in a CSV-file[5] to demonstrate some typical operations you will want to carry out with data stored in data bases. In most cases your triangles will be stored in tables and not in a classical triangle shape. The `ChainLadder` package contains a CSV-file with sample data in a long table format. We read the data into R's memory with the `read.csv` command and look at the first couple of rows and summarise it:

```
R> filename <-  file.path(system.file("Database",
+                                     package="ChainLadder"),
+                         "TestData.csv")
R> myData <- read.csv(filename)
R> head(myData)
```

```
  origin dev  value lob
1   1977   1 153638 ABC
2   1978   1 178536 ABC
3   1979   1 210172 ABC
4   1980   1 211448 ABC
5   1981   1 219810 ABC
6   1982   1 205654 ABC
```

```
R> summary(myData)
```

```
     origin          dev             value                            lob
 Min.   :   1   Min.   : 1.00   Min.   : -17657   AutoLiab           :105
 1st Qu.:   3   1st Qu.: 2.00   1st Qu.:   10324   GeneralLiab        :105
 Median :   6   Median : 4.00   Median :   72468   M3IR5              :105
```

[4] See the `RODBC` package

[5] Please ensure that your CSV-file is free from formatting, e.g. characters to separate units of thousands, as those columns will be read as characters or factors rather than numerical values.

```
 Mean   : 642   Mean   : 4.61   Mean   : 176632   ABC               : 66
 3rd Qu.:1979   3rd Qu.: 7.00   3rd Qu.: 197716   CommercialAutoPaid: 55
 Max.   :1991   Max.   :14.00   Max.   :3258646   GenIns            : 55
                                                  (Other)           :210
```

Let's focus on one subset of the data. We select the RAA data again:

```
R> raa <- subset(myData, lob %in% "RAA")
R> head(raa)
```

```
   origin dev value lob
67   1981   1  5012 RAA
68   1982   1   106 RAA
69   1983   1  3410 RAA
70   1984   1  5655 RAA
71   1985   1  1092 RAA
72   1986   1  1513 RAA
```

To transform the long table of the RAA data into a triangle we use the function `as.triangle`. The arguments we have to specify are the column names of the origin and development period and further the column which contains the values:

```
R> raa.tri <- as.triangle(raa,
+                         origin="origin",
+                         dev="dev",
+                         value="value")
R> raa.tri
```

```
      dev
origin    1    2    3    4    5    6    7   8   9  10
  1981 5012 3257 2638  898 1734 2642 1828 599  54 172
  1982  106 4179 1111 5270 3116 1817 -103 673 535  NA
  1983 3410 5582 4881 2268 2594 3479  649 603  NA  NA
  1984 5655 5900 4211 5500 2159 2658  984  NA  NA  NA
  1985 1092 8473 6271 6333 3786  225   NA  NA  NA  NA
  1986 1513 4932 5257 1233 2917   NA   NA  NA  NA  NA
  1987  557 3463 6926 1368   NA   NA   NA  NA  NA  NA
  1988 1351 5596 6165   NA   NA   NA   NA  NA  NA  NA
  1989 3133 2262   NA   NA   NA   NA   NA  NA  NA  NA
  1990 2063   NA   NA   NA   NA   NA   NA  NA  NA  NA
```

We note that the data has been stored as an incremental data set. As mentioned above, we could now use the function `incr2cum` to transform the triangle into a cumulative format.

We can transform a triangle back into a data frame structure:

```
R> raa.df <- as.data.frame(raa.tri, na.rm=TRUE)
R> head(raa.df)
```

```
       origin dev value
1981-1   1981   1  5012
1982-1   1982   1   106
1983-1   1983   1  3410
1984-1   1984   1  5655
1985-1   1985   1  1092
1986-1   1986   1  1513
```

This is particularly helpful when you would like to store your results back into a data base. Figure 3 gives you an idea of a potential data flow between R and data bases.

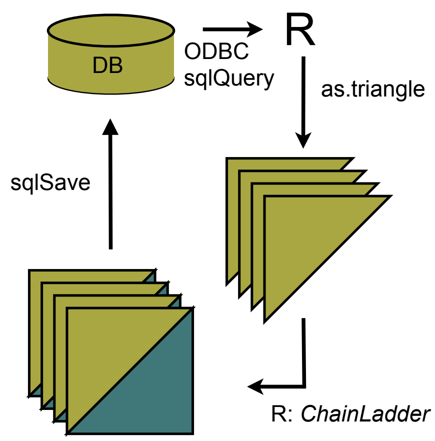


Figure 3: Flow chart of data between R and data bases.

3.1.4 Copying and pasting from MS Excel

Small data sets in Excel can be transfered to R backwards and forwards with via the clipboard under MS Windows.

Copying from Excel to R Select a data set in Excel and copy it into the clipboard, then go to R and type:

```
R> x <- read.table(file="clipboard", sep="\t", na.strings="")
```

Copying from R to Excel Suppose you would like to copy the RAA triangle into Excel, then the following statement would copy the data into the clipboard:

```
R> write.table(RAA, file="clipboard", sep="\t", na="")
```

Now you can paste the content into Excel. Please note that you can't copy lists structures from R to Excel.

3.2 Chain-ladder methods

The classical chain-ladder is a deterministic algorithm to forecast claims based on historical data. It assumes that the proportional developments of claims from one development period to the next are the same for all origin years.

3.2.1 Basic idea

Most commonly as a first step, the age-to-age link ratios are calculated as the volume weighted average development ratios of a cumulative loss development triangle from one development period to the next $C_{ik}, i, k = 1, \dots, n$.

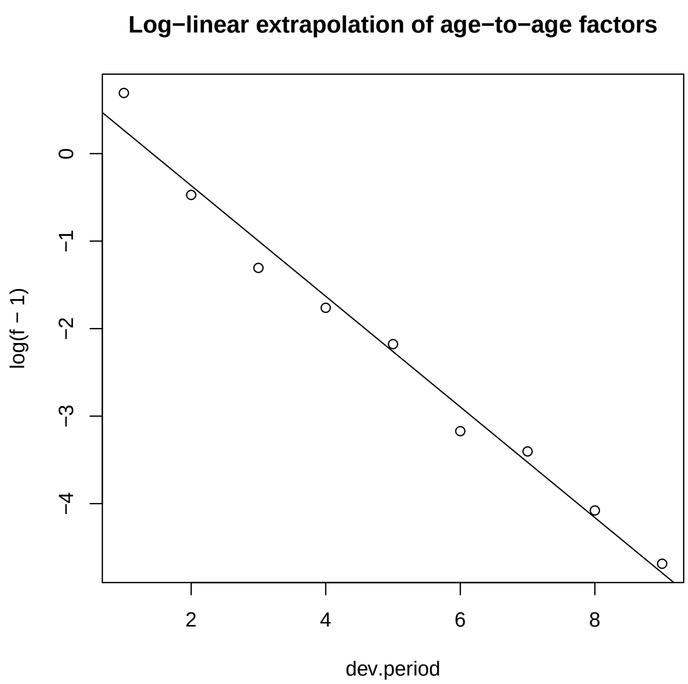
$$f_k = \frac{\sum_{i=1}^{n-k} C_{i,k+1}}{\sum_{i=1}^{n-k} C_{i,k}} \tag{1}$$

```
R> n <- 10
R> f <- sapply(1:(n-1),
+              function(i){
+                sum(RAA[c(1:(n-i)),i+1])/sum(RAA[c(1:(n-i)),i])
+              }
+              )
R> f

[1] 2.999 1.624 1.271 1.172 1.113 1.042 1.033 1.017 1.009
```

Often it is not suitable to assume that the oldest origin year is fully developed. A typical approach is to extrapolate the development ratios, e.g. assuming a log-linear model.

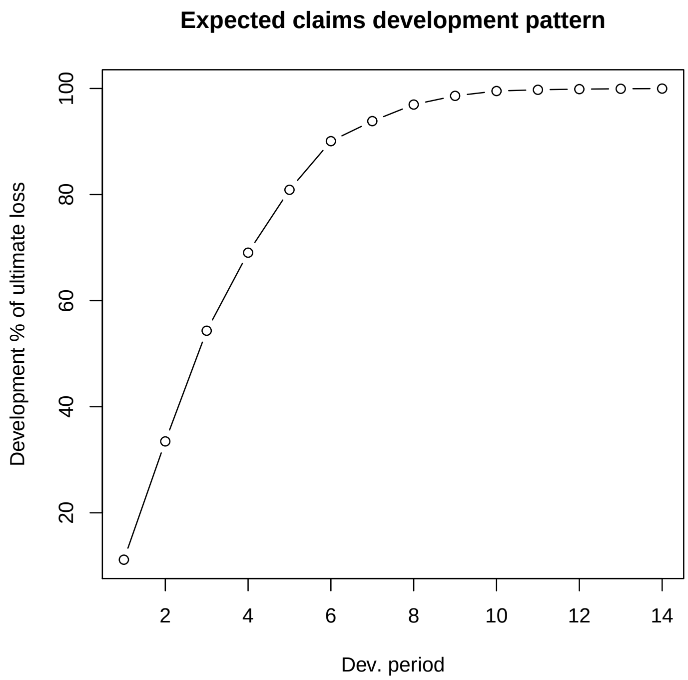
```
R> dev.period <- 1:(n-1)
R> plot(log(f-1) ~ dev.period, main="Log-linear extrapolation of age-to-age factors")
R> tail.model <- lm(log(f-1) ~ dev.period)
R> abline(tail.model)
R> co <- coef(tail.model)
R> ## extrapolate another 100 dev. period
```

```
R> tail <- exp(co[1] + c((n + 1):(n + 100)) * co[2]) + 1
R> f.tail <- prod(tail)
R> f.tail

[1] 1.005
```



The age-to-age factors allow us to plot the expected claims development patterns.

```
R> plot(100*(rev(1/cumprod(rev(c(f, tail[tail>1.0001]))))), t="b",
+       main="Expected claims development pattern",
+       xlab="Dev. period", ylab="Development % of ultimate loss")
```



The link ratios are then applied to the latest known cumulative claims amount to forecast the next development period. The *squaring* of the RAA triangle is calculated below, where an *ultimate* column is appended to the right to accommodate the expected development beyond the oldest age (10) of the triangle due to the tail factor (1.005) being greater than unity.

```
R> f <- c(f, f.tail)
R> fullRAA <- cbind(RAA, Ult = rep(0, 10))
R> for(k in 1:n){
+   fullRAA[(n-k+1):n, k+1] <- fullRAA[(n-k+1):n,k]*f[k]
+ }
R> round(fullRAA)
```

	1	2	3	4	5	6	7	8	9	10	Ult
1981	5012	8269	10907	11805	13539	16181	18009	18608	18662	18834	18928
1982	106	4285	5396	10666	13782	15599	15496	16169	16704	16858	16942
1983	3410	8992	13873	16141	18735	22214	22863	23466	23863	24083	24204
1984	5655	11555	15766	21266	23425	26083	27067	27967	28441	28703	28847
1985	1092	9565	15836	22169	25955	26180	27278	28185	28663	28927	29072
1986	1513	6445	11702	12935	15852	17649	18389	19001	19323	19501	19599
1987	557	4020	10946	12314	14428	16064	16738	17294	17587	17749	17838
1988	1351	6947	13112	16664	19525	21738	22650	23403	23800	24019	24139

```
1989 3133  5395  8759 11132 13043 14521 15130 15634 15898 16045 16125
1990 2063  6188 10046 12767 14959 16655 17353 17931 18234 18402 18495
```

The total estimated outstanding loss under this method is about 53200:

```
R> sum(fullRAA[ ,11] - getLatestCumulative(RAA))

[1] 53202
```

This approach is also called Loss Development Factor (LDF) method.

More generally, the factors used to square the triangle need not always be drawn from the dollar weighted averages of the triangle. Other sources of factors from which the actuary may *select* link ratios include simple averages from the triangle, averages weighted toward more recent observations or adjusted for outliers, and benchmark patterns based on related, more credible loss experience. Also, since the ultimate value of claims is simply the product of the most current diagonal and the cumulative product of the link ratios, the completion of interior of the triangle is usually not displayed in favor of that multiplicative calculation.

For example, suppose the actuary decides that the volume weighted factors from the RAA triangle are representative of expected future growth, but discards the 1.005 tail factor derived from the loglinear fit in favor of a five percent tail (1.05) based on loss data from a larger book of similar business. The LDF method might be displayed in R as follows.

```
R> linkratios <- c(attr(ata(RAA), "vwtd"), tail = 1.05)
R> round(linkratios, 3) # display to only three decimal places

  1-2   2-3   3-4   4-5   5-6   6-7   7-8   8-9  9-10  tail
2.999 1.624 1.271 1.172 1.113 1.042 1.033 1.017 1.009 1.050

R> LDF <- rev(cumprod(rev(linkratios)))
R> names(LDF) <- colnames(RAA) # so the display matches the triangle
R> round(LDF, 3)

    1     2     3     4     5     6     7     8     9    10
9.366 3.123 1.923 1.513 1.292 1.160 1.113 1.078 1.060 1.050

R> currentEval <- getLatestCumulative(RAA)
R> # Reverse the LDFs so the first, least mature factor [1]
R> #        is applied to the last origin year (1990)
R> EstdUlt <- currentEval * rev(LDF) #
R> # Start with the body of the exhibit
R> Exhibit <- data.frame(currentEval, LDF = round(rev(LDF), 3), EstdUlt)
```

```
R> # Tack on a Total row
R> Exhibit <- rbind(Exhibit,
+ data.frame(currentEval=sum(currentEval), LDF=NA, EstdUlt=sum(EstdUlt),
+            row.names = "Total"))
R> Exhibit
```

```
      currentEval   LDF EstdUlt
1981        18834 1.050   19776
1982        16704 1.060   17701
1983        23466 1.078   25288
1984        27067 1.113   30138
1985        26180 1.160   30373
1986        15852 1.292   20476
1987        12314 1.513   18637
1988        13112 1.923   25220
1989         5395 3.123   16847
1990         2063 9.366   19323
Total      160987    NA  223778
```

Since the early 1990s several papers have been published to embed the simple chain-ladder method into a statistical framework. Ben Zehnwirth and Glenn Barnett point out in [ZB00] that the age-to-age link ratios can be regarded as the coefficients of a weighted linear regression through the origin, see also [Mur94].

```
R> lmCL <- function(i, Triangle){
+   lm(y~x+0, weights=1/Triangle[,i],
+      data=data.frame(x=Triangle[,i], y=Triangle[,i+1]))
+ }
R> sapply(lapply(c(1:(n-1)), lmCL, RAA), coef)
```

```
    x     x     x     x     x     x     x     x     x
2.999 1.624 1.271 1.172 1.113 1.042 1.033 1.017 1.009
```

3.2.2 Mack chain-ladder

Thomas Mack published in 1993 [Mac93b] a method which estimates the standard errors of the chain-ladder forecast without assuming a distribution under three conditions.

Following the notation of Mack [Mac99] let C_{ik} denote the cumulative loss amounts of origin period (e.g. accident year) $i = 1, \dots, m$, with losses known for development period (e.g. development year) $k \leq n + 1 - i$.

In order to forecast the amounts C_{ik} for $k > n+1-i$ the Mack chain-ladder-model assumes:

CL1: $E[F_{ik}|C_{i1}, C_{i2}, \dots, C_{ik}] = f_k$ with $F_{ik} = \dfrac{C_{i,k+1}}{C_{ik}}$ (2)

CL2: $Var(\dfrac{C_{i,k+1}}{C_{ik}}|C_{i1}, C_{i2}, \dots, C_{ik}) = \dfrac{\sigma_k^2}{w_{ik} C_{ik}^{\alpha}}$ (3)

CL3: $\{C_{i1}, \dots, C_{in}\}, \{C_{j1}, \dots, C_{jn}\},$ are independent for origin period $i \neq j$ (4)

with $w_{ik} \in [0;1], \alpha \in \{0, 1, 2\}$. If these assumptions hold, the Mack-chain-ladder-model gives an unbiased estimator for IBNR (Incurred But Not Reported) claims.

The Mack-chain-ladder model can be regarded as a weighted linear regression through the origin for each development period: `lm(y ~ x  + 0, weights=w/x^(2-alpha))`, where `y` is the vector of claims at development period $k+1$ and `x` is the vector of claims at development period k.

The Mack method is implemented in the `ChainLadder` package via the function `MackChainLadder`.

As an example we apply the `MackChainLadder` function to our triangle `RAA`:

```
R> mack <- MackChainLadder(RAA, est.sigma="Mack")
R> mack

MackChainLadder(Triangle = RAA, est.sigma = "Mack")

     Latest Dev.To.Date Ultimate   IBNR Mack.S.E CV(IBNR)
1981 18,834       1.000   18,834      0        0      NaN
1982 16,704       0.991   16,858    154      206    1.339
1983 23,466       0.974   24,083    617      623    1.010
1984 27,067       0.943   28,703  1,636      747    0.457
1985 26,180       0.905   28,927  2,747    1,469    0.535
1986 15,852       0.813   19,501  3,649    2,002    0.549
1987 12,314       0.694   17,749  5,435    2,209    0.406
1988 13,112       0.546   24,019 10,907    5,358    0.491
1989  5,395       0.336   16,045 10,650    6,333    0.595
1990  2,063       0.112   18,402 16,339   24,566    1.503

               Totals
Latest:    160,987.00
Dev:             0.76
Ultimate:  213,122.23
IBNR:       52,135.23
Mack S.E.:  26,909.01
CV(IBNR):        0.52
```

We can access the loss development factors and the full triangle via

```
R> mack$f
```

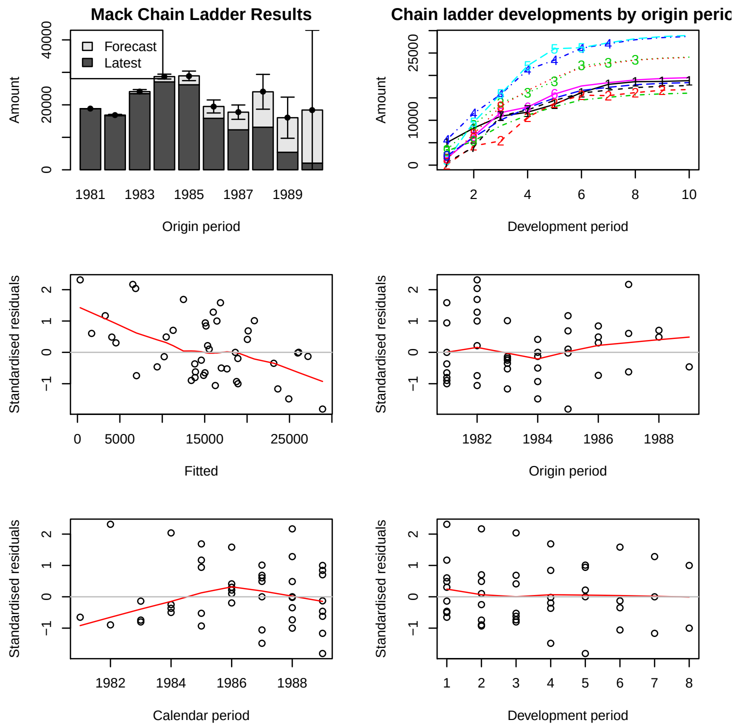
```
 [1] 2.999 1.624 1.271 1.172 1.113 1.042 1.033 1.017 1.009 1.000
```

```
R> mack$FullTriangle
```

```
      dev
origin    1     2     3     4     5     6     7     8     9    10
  1981 5012  8269 10907 11805 13539 16181 18009 18608 18662 18834
  1982  106  4285  5396 10666 13782 15599 15496 16169 16704 16858
  1983 3410  8992 13873 16141 18735 22214 22863 23466 23863 24083
  1984 5655 11555 15766 21266 23425 26083 27067 27967 28441 28703
  1985 1092  9565 15836 22169 25955 26180 27278 28185 28663 28927
  1986 1513  6445 11702 12935 15852 17649 18389 19001 19323 19501
  1987  557  4020 10946 12314 14428 16064 16738 17294 17587 17749
  1988 1351  6947 13112 16664 19525 21738 22650 23403 23800 24019
  1989 3133  5395  8759 11132 13043 14521 15130 15634 15898 16045
  1990 2063  6188 10046 12767 14959 16655 17353 17931 18234 18402
```

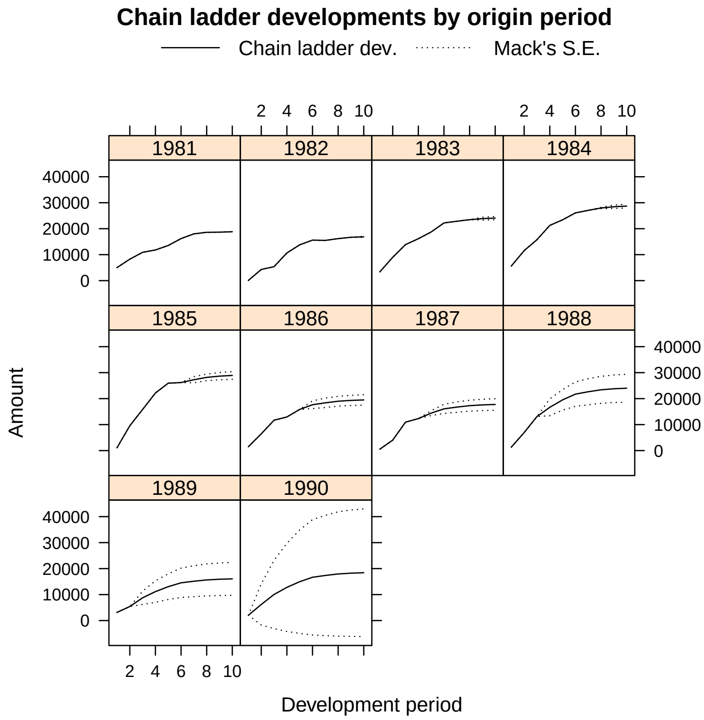
To check that Mack's assumption are valid review the residual plots, you should see no trends in either of them.

```
R> plot(mack)
```



We can plot the development, including the forecast and estimated standard errors by origin period by setting the argument `lattice=TRUE`.

```
R> plot(mack, lattice=TRUE)
```



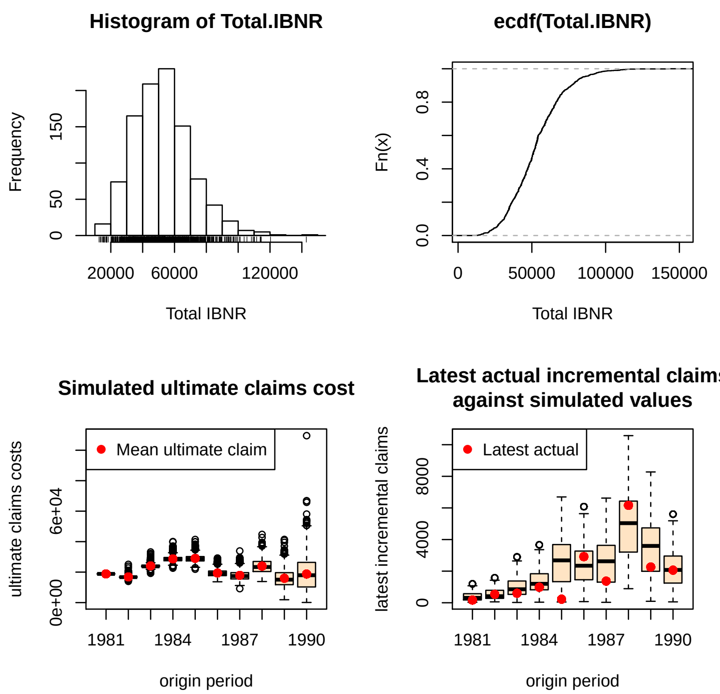
3.2.3 Bootstrap chain-ladder

The `BootChainLadder` function uses a two-stage bootstrapping/simulation approach following the paper by England and Verrall [PR02]. In the first stage an ordinary chain-ladder methods is applied to the cumulative claims triangle. From this we calculate the scaled Pearson residuals which we bootstrap R times to forecast future incremental claims payments via the standard chain-ladder method. In the second stage we simulate the process error with the bootstrap value as the mean and using the process distribution assumed. The set of reserves obtained in this way forms the predictive distribution, from which summary statistics such as mean, prediction error or quantiles can be derived.

```
R> ## See also the example in section 8 of England & Verrall (2002)
R> ## on page 55.
R> B <- BootChainLadder(RAA, R=999, process.distr="gamma")
R> B

BootChainLadder(Triangle = RAA, R = 999, process.distr = "gamma")

     Latest Mean Ultimate Mean IBNR SD IBNR IBNR 75% IBNR 95%
1981 18,834        18,834         0       0        0        0
```

```
1982 16,704         16,859      155    705      188    1,216
1983 23,466         24,105      639  1,334    1,073    3,096
1984 27,067         28,739    1,672  1,933    2,555    5,164
1985 26,180         29,025    2,845  2,360    4,023    7,095
1986 15,852         19,428    3,576  2,376    4,862    7,984
1987 12,314         17,894    5,580  3,143    7,340   11,274
1988 13,112         23,948   10,836  4,860   13,894   19,361
1989  5,395         15,957   10,562  5,943   14,254   21,982
1990  2,063         18,759   16,696 12,904   24,034   40,036

                  Totals
Latest:          160,987
Mean Ultimate:   213,547
Mean IBNR:        52,560
SD IBNR:          18,314
Total IBNR 75%:   63,523
Total IBNR 95%:   84,530
```

R> plot(B)

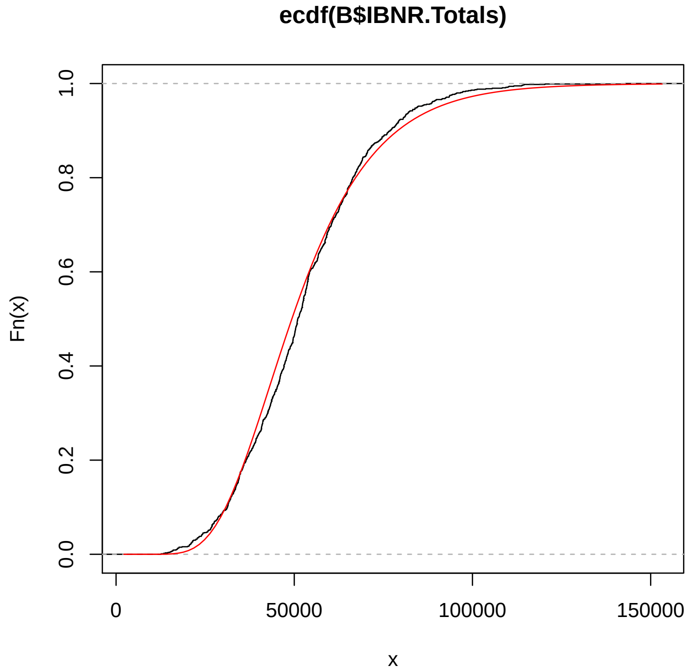


Quantiles of the bootstrap IBNR can be calculated via the `quantile` function:

```
R> quantile(B, c(0.75,0.95,0.99, 0.995))

$ByOrigin
     IBNR 75% IBNR 95% IBNR 99% IBNR 99.5%
1981      0.0        0        0          0
1982    188.5     1216     2697       3663
1983   1073.3     3096     5288       6170
1984   2554.8     5164     8048       9434
1985   4022.9     7095    10795      11409
1986   4862.4     7984    10589      11213
1987   7339.7    11274    15001      15931
1988  13893.7    19361    24320      25644
1989  14253.8    21982    27352      28882
1990  24033.7    40036    49932      52418

$Totals
            Totals
IBNR 75%:    63523
IBNR 95%:    84530
IBNR 99%:   105582
IBNR 99.5%: 111633
```

The distribution of the IBNR appears to follow a log-normal distribution, so let's fit it:

```
R> ## fit a distribution to the IBNR
R> library(MASS)
R> plot(ecdf(B$IBNR.Totals))
R> ## fit a log-normal distribution
R> fit <- fitdistr(B$IBNR.Totals[B$IBNR.Totals>0], "lognormal")
R> fit

    meanlog      sdlog
  10.806119    0.367667
 ( 0.011632) ( 0.008225)

R> curve(plnorm(x,fit$estimate["meanlog"], fit$estimate["sdlog"]),
+       col="red", add=TRUE)
```



3.2.4 Munich chain-ladder

The Mack-chain-ladder model forecasts future claims developments based on a historical cumulative claims development triangle and estimates the standard error around those [QM04].

```
R> MCLpaid
```

```
      dev
origin    1    2    3    4    5    6    7
     1  576 1804 1970 2024 2074 2102 2131
     2  866 1948 2162 2232 2284 2348   NA
     3 1412 3758 4252 4416 4494   NA   NA
     4 2286 5292 5724 5850   NA   NA   NA
     5 1868 3778 4648   NA   NA   NA   NA
     6 1442 4010   NA   NA   NA   NA   NA
     7 2044   NA   NA   NA   NA   NA   NA
```

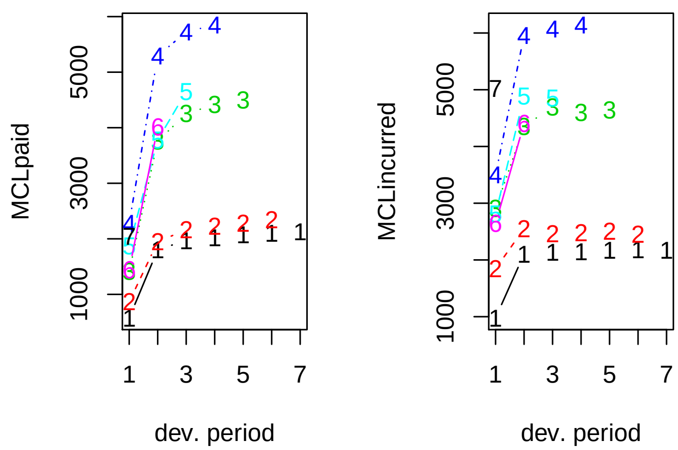
```
R> MCLincurred
```

```
      dev
origin    1    2    3    4    5    6    7
     1  978 2104 2134 2144 2174 2182 2174
     2 1844 2552 2466 2480 2508 2454   NA
     3 2904 4354 4698 4600 4644   NA   NA
     4 3502 5958 6070 6142   NA   NA   NA
     5 2812 4882 4852   NA   NA   NA   NA
     6 2642 4406   NA   NA   NA   NA   NA
     7 5022   NA   NA   NA   NA   NA   NA
```

```
R> op <- par(mfrow=c(1,2))
R> plot(MCLpaid)
R> plot(MCLincurred)
R> par(op)
R> # Following the example in Quarg's (2004) paper:
R> MCL <- MunichChainLadder(MCLpaid, MCLincurred, est.sigmaP=0.1, est.sigmaI=0.1)
R> MCL
```

```
MunichChainLadder(Paid = MCLpaid, Incurred = MCLincurred, est.sigmaP = 0.1,
    est.sigmaI = 0.1)

  Latest Paid Latest Incurred Latest P/I Ratio Ult. Paid Ult. Incurred
1       2,131           2,174            0.980     2,131         2,174
2       2,348           2,454            0.957     2,383         2,444
3       4,494           4,644            0.968     4,597         4,629
4       5,850           6,142            0.952     6,119         6,176
5       4,648           4,852            0.958     4,937         4,950
6       4,010           4,406            0.910     4,656         4,665
7       2,044           5,022            0.407     7,549         7,650
  Ult. P/I Ratio
1          0.980
2          0.975
3          0.993
4          0.991
5          0.997
6          0.998
7          0.987

Totals
            Paid Incurred P/I Ratio
Latest:   25,525   29,694      0.86
Ultimate: 32,371   32,688      0.99
```

```
R> plot(MCL)
```



3.3 Multivariate chain-ladder

The Mack chain ladder technique can be generalized to the multivariate setting where multiple reserving triangles are modeled and developed simultaneously. The advantage of the multivariate modeling is that correlations among different triangles can be modeled, which will lead to more accurate uncertainty assessments. Reserving methods that explicitly model the between-triangle contemporaneous correlations can be found in [PS05, MW08]. Another benefit of multivariate loss reserving is that structural relationships between triangles can also be reflected, where the development of one triangle depends on past losses from other triangles. For example, there is generally need for the joint development of the paid and incurred losses [QM04]. Most of the chain-ladder-based multivariate reserving models can be summarised as sequential seemingly unrelated regressions [Zha10]. We note another strand of multivariate loss reserving builds a hierarchical structure into the model to allow estimation of one triangle to "borrow strength" from other triangles, reflecting the core insight of actuarial credibility [ZDG12].

Denote $Y_{i,k} = (Y_{i,k}^{(1)}, \cdots, Y_{i,k}^{(N)})$ as an $N \times 1$ vector of cumulative losses at accident year i and development year k where (n) refers to the n-th triangle. [Zha10] specifies the model in development period k as:

$$Y_{i,k+1} = A_k + B_k \cdot Y_{i,k} + \epsilon_{i,k}, \tag{5}$$

where A_k is a column of intercepts and B_k is the development matrix for development period k. Assumptions for this model are:

$$E(\epsilon_{i,k}|Y_{i,1}, \cdots, Y_{i,I+1-k}) = 0. \tag{6}$$

$$cov(\epsilon_{i,k}|Y_{i,1}, \cdots, Y_{i,I+1-k}) = D(Y_{i,k}^{-\delta/2})\Sigma_k D(Y_{i,k}^{-\delta/2}). \tag{7}$$

losses of different accident years are independent. (8)

$\epsilon_{i,k}$ are symmetrically distributed. (9)

In the above, D is the diagonal operator, and δ is a known positive value that controls how the variance depends on the mean (as weights). This model is referred to as the general multivariate chain ladder [GMCL] in [Zha10]. A important special case where $A_k = 0$ and B_k's are diagonal is a naive generalization of the chain ladder, often referred to as the multivariate chain ladder [MCL] [PS05].

In the following, we first introduce the class "triangles", for which we have defined several utility functions. Indeed, any input triangles to the MultiChainLadder function will be converted to "triangles" internally. We then present loss reserving methods based on the MCL and GMCL models in turn.

3.3.1 The "triangles" class

Consider the two liability loss triangles from [MW08]. It comes as a list of two matrices :

```
R> str(liab)
```

```
List of 2
 $ GeneralLiab: num [1:14, 1:14] 59966 49685 51914 84937 98921 ...
 $ AutoLiab   : num [1:14, 1:14] 114423 152296 144325 145904 170333 ...
```

We can convert a list to a "triangles" object using

```
R> liab2 <- as(liab, "triangles")
R> class(liab2)
```

```
[1] "triangles"
attr(,"package")
[1] "ChainLadder"
```

We can find out what methods are available for this class:

```
R> showMethods(classes = "triangles")
```

For example, if we want to extract the last three columns of each triangle, we can use the "[" operator as follows:

```
R> # use drop = TRUE to remove rows that are all NA's
R> liab2[, 12:14, drop = TRUE]

An object of class "triangles"
[[1]]
       [,1]   [,2]   [,3]
[1,] 540873 547696 549589
[2,] 563571 562795     NA
[3,] 602710     NA     NA

[[2]]
       [,1]   [,2]   [,3]
[1,] 391328 391537 391428
[2,] 485138 483974     NA
[3,] 540742     NA     NA
```

The following combines two columns of the triangles to form a new matrix:

```
R> cbind2(liab2[1:3, 12])

       [,1]   [,2]
[1,] 540873 391328
[2,] 563571 485138
[3,] 602710 540742
```

3.3.2 Separate chain ladder ignoring correlations

The form of regression models used in estimating the development parameters is controlled by the `fit.method` argument. If we specify `fit.method = "OLS"`, the ordinary least squares will be used and the estimation of development factors for each triangle is independent of the others. In this case, the residual covariance matrix Σ_k is diagonal. As a result, the multivariate model is equivalent to running multiple Mack chain ladders separately.

```
R> fit1 <- MultiChainLadder(liab, fit.method = "OLS")
R> lapply(summary(fit1)$report.summary, "[", 15, )

$`Summary Statistics for Triangle 1`
        Latest Dev.To.Date Ultimate    IBNR    S.E     CV
Total 11343397      0.6482 17498658 6155261 427289 0.0694

$`Summary Statistics for Triangle 2`
       Latest Dev.To.Date Ultimate    IBNR    S.E     CV
Total 8759806      0.8093 10823418 2063612 162872 0.0789
```

```
$`Summary Statistics for Triangle 1+2`
        Latest Dev.To.Date Ultimate    IBNR    S.E     CV
Total 20103203      0.7098 28322077 8218874 457278 0.0556
```

In the above, we only show the total reserve estimate for each triangle to reduce the output. The full summary including the estimate for each year can be retrieved using the usual `summary` function. By default, the `summary` function produces reserve statistics for all individual triangles, as well as for the portfolio that is assumed to be the sum of the two triangles. This behavior can be changed by supplying the `portfolio` argument. See the documentation for details.

We can verify if this is indeed the same as the univariate Mack chain ladder. For example, we can apply the `MackChainLadder` function to each triangle:

```
R> fit <- lapply(liab, MackChainLadder, est.sigma = "Mack")
R> # the same as the first triangle above
R> lapply(fit, function(x) t(summary(x)$Totals))
```

```
$GeneralLiab
        Latest:   Dev: Ultimate:   IBNR: Mack S.E.: CV(IBNR):
Totals 11343397 0.6482  17498658 6155261     427289   0.06942

$AutoLiab
       Latest:   Dev: Ultimate:   IBNR: Mack S.E.: CV(IBNR):
Totals 8759806 0.8093  10823418 2063612     162872   0.07893
```

The argument `mse.method` controls how the mean square errors are computed. By default, it implements the Mack method. An alternative method is the conditional re-sampling approach in [BBMW06], which assumes the estimated parameters are independent. This is used when `mse.method = "Independence"`. For example, the following reproduces the result in [BBMW06]. Note that the first argument must be a list, even though only one triangle is used.

```
R> (B1 <- MultiChainLadder(list(GenIns), fit.method = "OLS",
+     mse.method = "Independence"))
```

```
$`Summary Statistics for Input Triangle`
          Latest Dev.To.Date  Ultimate      IBNR       S.E    CV
1      3,901,463      1.0000 3,901,463         0         0 0.000
2      5,339,085      0.9826 5,433,719    94,634    75,535 0.798
3      4,909,315      0.9127 5,378,826   469,511   121,700 0.259
4      4,588,268      0.8661 5,297,906   709,638   133,551 0.188
5      3,873,311      0.7973 4,858,200   984,889   261,412 0.265
6      3,691,712      0.7223 5,111,171 1,419,459   411,028 0.290
```

```
7      3,483,130      0.6153  5,660,771  2,177,641   558,356 0.256
8      2,864,498      0.4222  6,784,799  3,920,301   875,430 0.223
9      1,363,294      0.2416  5,642,266  4,278,972   971,385 0.227
10       344,014      0.0692  4,969,825  4,625,811 1,363,385 0.295
Total 34,358,090      0.6478 53,038,946 18,680,856 2,447,618 0.131
```

3.3.3 Multivariate chain ladder using seemingly unrelated regressions

To allow correlations to be incorporated, we employ the seemingly unrelated regressions (see the package `systemfit`) that simultaneously model the two triangles in each development period. This is invoked when we specify `fit.method = "SUR"`:

```
R> fit2 <- MultiChainLadder(liab, fit.method = "SUR")
R> lapply(summary(fit2)$report.summary, "[", 15, )

$`Summary Statistics for Triangle 1`
        Latest Dev.To.Date Ultimate    IBNR    S.E     CV
Total 11343397      0.6484 17494907 6151510 419293 0.0682

$`Summary Statistics for Triangle 2`
       Latest Dev.To.Date Ultimate    IBNR    S.E     CV
Total 8759806      0.8095 10821341 2061535 162464 0.0788

$`Summary Statistics for Triangle 1+2`
        Latest Dev.To.Date Ultimate    IBNR    S.E    CV
Total 20103203        0.71 28316248 8213045 500607 0.061
```

We see that the portfolio prediction error is inflated to $500,607$ from $457,278$ in the separate development model ("OLS"). This is because of the positive correlation between the two triangles. The estimated correlation for each development period can be retrieved through the `residCor` function:

```
R> round(unlist(residCor(fit2)), 3)

 [1]  0.247  0.495  0.682  0.446  0.487  0.451 -0.172  0.805  0.337  0.688
[11] -0.004  1.000  0.021
```

Similarly, most methods that work for linear models such as `coef`, `fitted`, `resid` and so on will also work. Since we have a sequence of models, the retrieved results from these methods are stored in a list. For example, we can retrieve the estimated development factors for each period as

```
R> do.call("rbind", coef(fit2))
```

```
      eq1_x[[1]] eq2_x[[2]]
 [1,]      3.227     2.2224
 [2,]      1.719     1.2688
 [3,]      1.352     1.1200
 [4,]      1.179     1.0665
 [5,]      1.106     1.0356
 [6,]      1.055     1.0168
 [7,]      1.026     1.0097
 [8,]      1.015     1.0002
 [9,]      1.012     1.0038
[10,]      1.006     0.9994
[11,]      1.005     1.0039
[12,]      1.005     0.9989
[13,]      1.003     0.9997
```

The smaller-than-one development factors after the 10-th period for the second triangle indeed result in negative IBNR estimates for the first several accident years in that triangle.

The package also offers the `plot` method that produces various summary and diagnostic figures:

```
R> parold <- par(mfrow = c(4, 2), mar = c(4, 4, 2, 1),
+     mgp = c(1.3, 0.3, 0), tck = -0.02)
R> plot(fit2, which.triangle = 1:2, which.plot = 1:4)
R> par(parold)
```

The resulting plots are shown in Figure 4. We use `which.triangle` to suppress the plot for the portfolio, and use `which.plot` to select the desired types of plots. See the documentation for possible values of these two arguments.

3.3.4 Other residual covariance estimation methods

Internally, the `MultiChainLadder` calls the `systemfit` function to fit the regression models period by period. When SUR models are specified, there are several ways to estimate the residual covariance matrix Σ_k. Available methods are `"noDfCor"`, `"geomean"`, `"max"`, and `"Theil"` with the default as `"geomean"`. The method `"Theil"` will produce unbiased covariance estimate, but the resulting estimate may not be positive semi-definite. This is also the estimator used by [MW08]. However, this method does not work out of the box for the `liab` data, and is perhaps one of the reasons [MW08] used extrapolation to get the estimate for the last several periods.

Indeed, for most applications, we recommend the use of separate chain ladders for the tail periods to stabilize the estimation - there are few data points in the tail and running a multivariate model often produces extremely volatile estimates or even

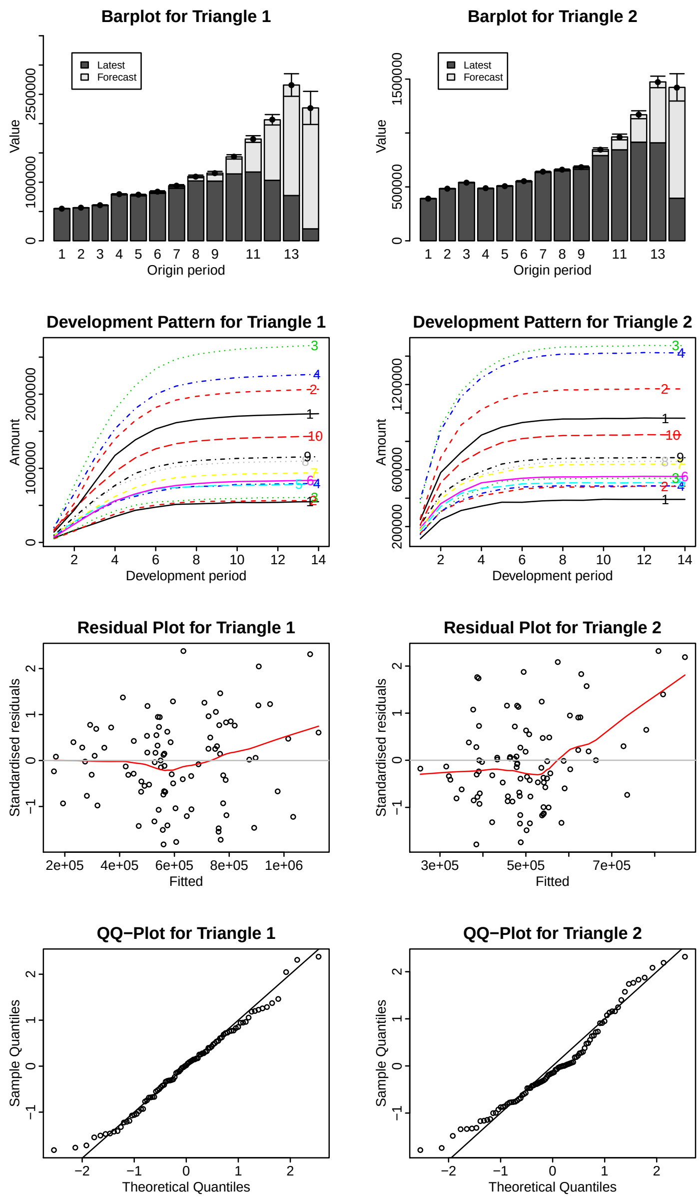


Figure 4: Summary and diagnostic plots from a `MultiChainLadder` object.

fails. To facilitate such an approach, the package offers the MultiChainLadder2 function, which implements a split-and-join procedure: we split the input data into two parts, specify a multivariate model with rich structures on the first part (with enough data) to reflect the multivariate dependencies, apply separate univariate chain ladders on the second part, and then join the two models together to produce the final predictions. The splitting is determined by the "last" argument, which specifies how many of the development periods in the tail go into the second part of the split. The type of the model structure to be specified for the first part of the split model in MultiChainLadder2 is controlled by the type argument. It takes one of the following values: "MCL"- the multivariate chain ladder with diagonal development matrix; "MCL+int"- the multivariate chain ladder with additional intercepts; "GMCL-int"- the general multivariate chain ladder without intercepts; and "GMCL" - the full general multivariate chain ladder with intercepts and non-diagonal development matrix.

For example, the following fits the SUR method to the first part (the first 11 columns) using the unbiased residual covariance estimator in [MW08], and separate chain ladders for the rest:

```
R> W1 <- MultiChainLadder2(liab, mse.method = "Independence",
+               control = systemfit.control(methodResidCov = "Theil"))
R> lapply(summary(W1)$report.summary, "[", 15, )

$`Summary Statistics for Triangle 1`
        Latest Dev.To.Date Ultimate    IBNR    S.E     CV
Total 11343397      0.6483 17497403 6154006 427041 0.0694

$`Summary Statistics for Triangle 2`
       Latest Dev.To.Date Ultimate    IBNR    S.E    CV
Total 8759806      0.8095 10821034 2061228 162785 0.079

$`Summary Statistics for Triangle 1+2`
        Latest Dev.To.Date Ultimate    IBNR    S.E     CV
Total 20103203      0.7099 28318437 8215234 505376 0.0615
```

Similary, the iterative residual covariance estimator in [MW08] can also be used, in which we use the control parameter maxiter to determine the number of iterations:

```
R> for (i in 1:5){
+   W2 <- MultiChainLadder2(liab, mse.method = "Independence",
+       control = systemfit.control(methodResidCov = "Theil", maxiter = i))
+   print(format(summary(W2)@report.summary[[3]][15, 4:5],
+           digits = 6, big.mark = ","))
+ }

           IBNR     S.E
Total 8,215,234 505,376
```

```
           IBNR     S.E
Total 8,215,357 505,443
           IBNR     S.E
Total 8,215,362 505,444
           IBNR     S.E
Total 8,215,362 505,444
           IBNR     S.E
Total 8,215,362 505,444
```

```
R> lapply(summary(W2)$report.summary, "[", 15, )
```

```
$`Summary Statistics for Triangle 1`
        Latest Dev.To.Date Ultimate    IBNR    S.E     CV
Total 11343397      0.6483 17497526 6154129 427074 0.0694

$`Summary Statistics for Triangle 2`
       Latest Dev.To.Date Ultimate    IBNR    S.E    CV
Total 8759806      0.8095 10821039 2061233 162790 0.079

$`Summary Statistics for Triangle 1+2`
        Latest Dev.To.Date Ultimate    IBNR    S.E     CV
Total 20103203      0.7099 28318565 8215362 505444 0.0615
```

We see that the covariance estimate converges in three steps. These are very similar to the results in [MW08], the small difference being a result of the different approaches used in the last three periods.

Also note that in the above two examples, the argument `control` is not defined in the proptotype of the `MultiChainLadder`. It is an argument that is passed to the `systemfit` function through the `...` mechanism. Users are encouraged to explore how other options available in `systemfit` can be applied.

3.3.5 Model with intercepts

Consider the `auto` triangles from [Zha10]. It includes three automobile insurance triangles: personal auto paid, personal auto incurred, and commercial auto paid.

```
R> str(auto)
```

```
List of 3
 $ PersonalAutoPaid    : num [1:10, 1:10] 101125 102541 114932 114452 115597 ...
 $ PersonalAutoIncurred: num [1:10, 1:10] 325423 323627 358410 405319 434065 ...
 $ CommercialAutoPaid  : num [1:10, 1:10] 19827 22331 22533 23128 25053 ...
```

It is a reasonable expectation that these triangles will be correlated. So we run a MCL model on them:

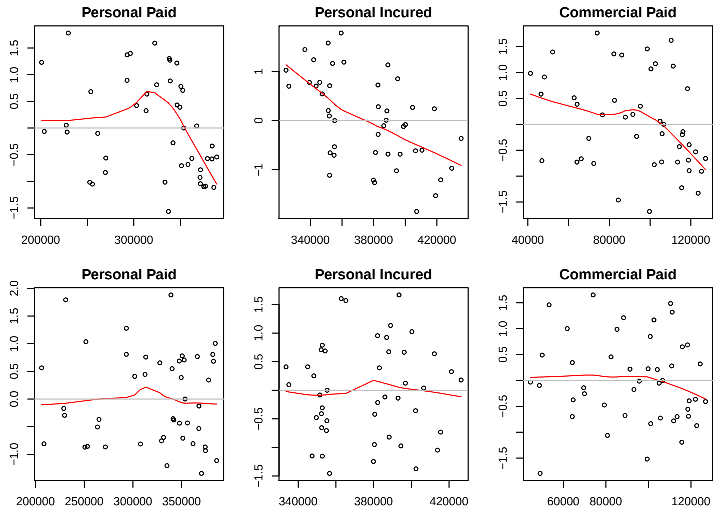


Figure 5: Residual plots for the MCL model (first row) and the GMCL (MCL+int) model (second row) for the auto data.

```
R> f0 <- MultiChainLadder2(auto, type = "MCL")
R> # show correlation- the last three columns have zero correlation
R> # because separate chain ladders are used
R> print(do.call(cbind, residCor(f0)), digits = 3)
```

```
       [,1]    [,2]  [,3]  [,4]    [,5]  [,6] [,7] [,8] [,9]
(1,2) 0.327 -0.0101 0.598 0.711  0.8565 0.928    0    0    0
(1,3) 0.870  0.9064 0.939 0.261 -0.0607 0.911    0    0    0
(2,3) 0.198 -0.3217 0.558 0.380  0.3586 0.931    0    0    0
```

However, from the residual plot, the first row in Figure 5, it is evident that the default mean structure in the MCL model is not adequate. Usually this is a common problem with the chain ladder based models, owing to the missing of intercepts.

We can improve the above model by including intercepts in the SUR fit as follows:

```
R> f1 <- MultiChainLadder2(auto, type = "MCL+int")
```

The corresponding residual plot is shown in the second row in Figure 5. We see that these residuals are randomly scattered around zero and there is no clear pattern compared to the plot from the MCL model.

The default summary computes the portfolio estimates as the sum of all the triangles. This is not desirable because the first two triangles are both from the personal auto line. We can overwrite this via the `portfolio` argument. For example, the following uses the two paid triangles as the portfolio estimate:

```
R> lapply(summary(f1, portfolio = "1+3")@report.summary, "[", 11, )

$`Summary Statistics for Triangle 1`
       Latest Dev.To.Date Ultimate   IBNR   S.E     CV
Total 3290539      0.8537  3854572 564033 19089 0.0338

$`Summary Statistics for Triangle 2`
       Latest Dev.To.Date Ultimate  IBNR   S.E     CV
Total 3710614      0.9884  3754197 43583 18839 0.4323

$`Summary Statistics for Triangle 3`
       Latest Dev.To.Date Ultimate   IBNR   S.E     CV
Total 1043851      0.7504  1391064 347213 27716 0.0798

$`Summary Statistics for Triangle 1+3`
       Latest Dev.To.Date Ultimate   IBNR   S.E     CV
Total 4334390      0.8263  5245636 911246 38753 0.0425
```

3.3.6 Joint modeling of the paid and incurred losses

Although the model with intercepts proved to be an improvement over the MCL model, it still fails to account for the structural relationship between triangles. In particular, it produces divergent paid-to-incurred loss ratios for the personal auto line:

```
R> ult <- summary(f1)$Ultimate
R> print(ult[, 1] /ult[, 2], 3)

    1     2     3     4     5     6     7     8     9    10 Total
0.995 0.995 0.993 0.992 0.995 0.996 1.021 1.067 1.112 1.114 1.027
```

We see that for accident years 9-10, the paid-to-incurred loss ratios are more than 110%. This can be fixed by allowing the development of the paid/incurred triangles to depend on each other. That is, we include the past values from the paid triangle as predictors when developing the incurred triangle, and vice versa.

We illustrate this ignoring the commercial auto triangle. See the demo for a model that uses all three triangles. We also include the MCL model and the Munich chain ladder as a comparison:

```
R> da <- auto[1:2]
R> # MCL with diagonal development
R> M0 <- MultiChainLadder(da)
R> # non-diagonal development matrix with no intercepts
R> M1 <- MultiChainLadder2(da, type = "GMCL-int")
```

```
R> # Munich Chain Ladder
R> M2 <- MunichChainLadder(da[[1]], da[[2]])
R> # compile results and compare projected paid to incured ratios
R> r1 <- lapply(list(M0, M1), function(x){
+          ult <- summary(x)@Ultimate
+          ult[, 1] / ult[, 2]
+      })
R> names(r1) <- c("MCL", "GMCL")
R> r2 <- summary(M2)[[1]][, 6]
R> r2 <- c(r2, summary(M2)[[2]][2, 3])
R> print(do.call(cbind, c(r1, list(MuCl = r2))) * 100, digits = 4)
```

```
         MCL   GMCL   MuCl
1      99.50  99.50  99.50
2      99.49  99.49  99.55
3      99.29  99.29 100.23
4      99.20  99.20 100.23
5      99.83  99.56 100.04
6     100.43  99.66 100.03
7     103.53  99.76  99.95
8     111.24 100.02  99.81
9     122.11 100.20  99.67
10    126.28 100.18  99.69
Total 105.58  99.68  99.88
```

3.4 Clark's methods

The ChainLadder package contains functionality to carry out the methods described in the paper [6] by David Clark [Cla03] . Using a longitudinal analysis approach, Clark assumes that losses develop according to a theoretical *growth curve*. The LDF method is a special case of this approach where the growth curve can be considered to be either a step function or piecewise linear. Clark envisions a growth curve as measuring the percent of ultimate loss that can be expected to have emerged as of each age of an origin period. The paper describes two methods that fit this model.

The LDF method assumes that the ultimate losses in each origin period are separate and unrelated. The goal of the method, therefore, is to estimate parameters for the ultimate losses and for the growth curve in order to maximize the likelihood of having observed the data in the triangle.

The CapeCod method assumes that the *apriori* expected ultimate losses in each origin year are the product of earned premium that year and a theoretical loss ratio. The CapeCod method, therefore, need estimate potentially far fewer parameters:

[6] This paper is on the CAS Exam 6 syllabus.

for the growth function and for the theoretical loss ratio.

One of the side benefits of using maximum likelihood to estimate parameters is that its associated asymptotic theory provides uncertainty estimates for the parameters. Observing that the reserve estimates by origin year are functions of the estimated parameters, uncertainty estimates of these functional values are calculated according to the *Delta method*, which is essentially a linearisation of the problem based on a Taylor series expansion.

The two functional forms for growth curves considered in Clark's paper are the loglogistic function (a.k.a., the inverse power curve) and the Weibull function, both being two-parameter functions. Clark uses the parameters ω and θ in his paper. Clark's methods work on incremental losses. His likelihood function is based on the assumption that incremental losses follow an over-dispersed Poisson (ODP) process.

3.4.1 Clark's LDF method

Consider again the RAA triangle. Accepting all defaults, the Clark LDF Method would estimate total ultimate losses of 272,009 and a reserve (FutureValue) of 111,022, or almost twice the value based on the volume weighted average link ratios and loglinear fit in section 3.2.1 above.

```
R> ClarkLDF(RAA)
```

```
 Origin CurrentValue    Ldf UltimateValue FutureValue StdError  CV%
   1981       18,834  1.216        22,906       4,072    2,792 68.6
   1982       16,704  1.251        20,899       4,195    2,833 67.5
   1983       23,466  1.297        30,441       6,975    4,050 58.1
   1984       27,067  1.360        36,823       9,756    5,147 52.8
   1985       26,180  1.451        37,996      11,816    5,858 49.6
   1986       15,852  1.591        25,226       9,374    4,877 52.0
   1987       12,314  1.829        22,528      10,214    5,206 51.0
   1988       13,112  2.305        30,221      17,109    7,568 44.2
   1989        5,395  3.596        19,399      14,004    7,506 53.6
   1990        2,063 12.394        25,569      23,506   17,227 73.3
  Total      160,987              272,009     111,022   36,102 32.5
```

Most of the difference is due to the heavy tail, 21.6%, implied by the inverse power curve fit. Clark recognizes that the log-logistic curve can take an unreasonably long length of time to flatten out. If according to the actuary's experience most claims close as of, say, 20 years, the growth curve can be truncated accordingly by using the `maxage` argument:

```
R> ClarkLDF(RAA, maxage = 20)
```

Origin	CurrentValue	Ldf	UltimateValue	FutureValue	StdError	CV%
1981	18,834	1.124	21,168	2,334	1,765	75.6
1982	16,704	1.156	19,314	2,610	1,893	72.6
1983	23,466	1.199	28,132	4,666	2,729	58.5
1984	27,067	1.257	34,029	6,962	3,559	51.1
1985	26,180	1.341	35,113	8,933	4,218	47.2
1986	15,852	1.471	23,312	7,460	3,775	50.6
1987	12,314	1.691	20,819	8,505	4,218	49.6
1988	13,112	2.130	27,928	14,816	6,300	42.5
1989	5,395	3.323	17,927	12,532	6,658	53.1
1990	2,063	11.454	23,629	21,566	15,899	73.7
Total	160,987		251,369	90,382	26,375	29.2

The Weibull growth curve tends to be faster developing than the log-logistic:

```
R> ClarkLDF(RAA, G="weibull")
```

Origin	CurrentValue	Ldf	UltimateValue	FutureValue	StdError	CV%
1981	18,834	1.022	19,254	420	700	166.5
1982	16,704	1.037	17,317	613	855	139.5
1983	23,466	1.060	24,875	1,409	1,401	99.4
1984	27,067	1.098	29,728	2,661	2,037	76.5
1985	26,180	1.162	30,419	4,239	2,639	62.2
1986	15,852	1.271	20,151	4,299	2,549	59.3
1987	12,314	1.471	18,114	5,800	3,060	52.8
1988	13,112	1.883	24,692	11,580	4,867	42.0
1989	5,395	2.988	16,122	10,727	5,544	51.7
1990	2,063	9.815	20,248	18,185	12,929	71.1
Total	160,987		220,920	59,933	19,149	32.0

It is recommend to inspect the residuals to help assess the reasonableness of the model relative to the actual data.

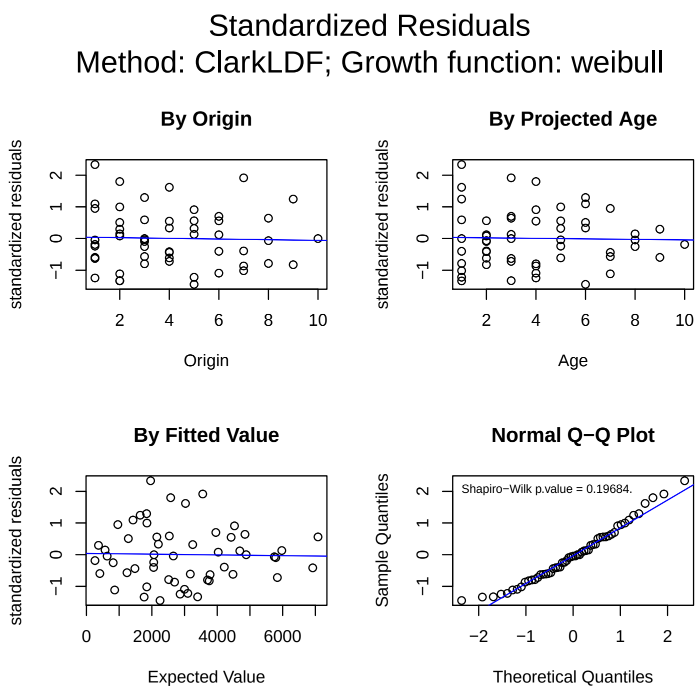
Although there is some evidence of heteroscedasticity with increasing ages and fitted values, the residuals otherwise appear randomly scattered around a horizontal line through the origin. The q-q plot shows evidence of a lack of fit in the tails, but the p-value of almost 0.2 can be considered too high to reject outright the assumption of normally distributed standardized residuals[7] .

3.4.2 Clark's Cap Cod method

The RAA data set, widely researched in the literature, has no premium associated with it traditionally. Let's assume a constant earned premium of 40000 each year, and a Weibull growth function:

[7] As an exercise, the reader can confirm that the normal distribution assumption is rejected at the 5% level with the log-logistic curve.

```
R> plot(ClarkLDF(RAA, G="weibull"))
```




```
R> ClarkCapeCod(RAA, Premium = 40000, G = "weibull")
```

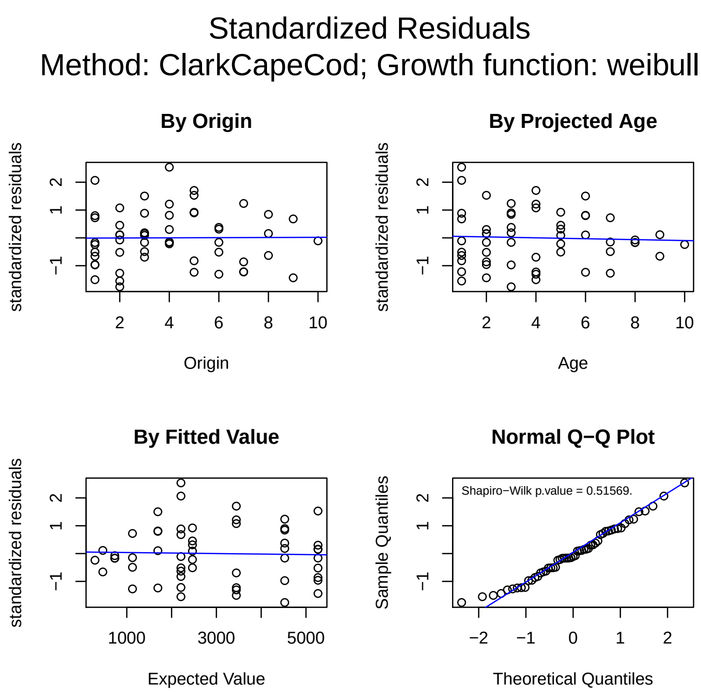
```
 Origin CurrentValue Premium   ELR FutureGrowthFactor FutureValue UltimateValue
   1981       18,834  40,000 0.566             0.0192         436        19,270
   1982       16,704  40,000 0.566             0.0320         725        17,429
   1983       23,466  40,000 0.566             0.0525       1,189        24,655
   1984       27,067  40,000 0.566             0.0848       1,921        28,988
   1985       26,180  40,000 0.566             0.1345       3,047        29,227
   1986       15,852  40,000 0.566             0.2093       4,741        20,593
   1987       12,314  40,000 0.566             0.3181       7,206        19,520
   1988       13,112  40,000 0.566             0.4702      10,651        23,763
   1989        5,395  40,000 0.566             0.6699      15,176        20,571
   1990        2,063  40,000 0.566             0.9025      20,444        22,507
  Total      160,987 400,000                               65,536       226,523
 StdError   CV%
      692 158.6
      912 125.7
    1,188  99.9
    1,523  79.3
```

```
 1,917  62.9
 2,360  49.8
 2,845  39.5
 3,366  31.6
 3,924  25.9
 4,491  22.0
12,713  19.4
```

The estimated expected loss ratio is 0.566. The total outstanding loss is about 10% higher than with the LDF method. The standard error, however, is lower, probably due to the fact that there are fewer parameters to estimate with the CapeCod method, resulting in less parameter risk.

A plot of this model shows similar residuals By Origin and Projected Age to those from the LDF method, a better spread By Fitted Value, and a slightly better q-q plot, particularly in the upper tail.

```
R> plot(ClarkCapeCod(RAA, Premium = 40000, G = "weibull"))
```



3.5 Generalised linear model methods

Recent years have also seen growing interest in using generalised linear models [GLM] for insurance loss reserving. The use of GLM in insurance loss reserving has many compelling aspects, e.g.,

- when over-dispersed Poisson model is used, it reproduces the estimates from Chain Ladder;
- it provides a more coherent modeling framework than the Mack method;
- all the relevant established statistical theory can be directly applied to perform hypothesis testing and diagnostic checking;

The `glmReserve` function takes an insurance loss triangle, converts it to incremental losses internally if necessary, transforms it to the long format (see `as.data.frame`) and fits the resulting loss data with a generalised linear model where the mean structure includes both the accident year and the development lag effects. The function also provides both analytical and bootstrapping methods to compute the associated prediction errors. The bootstrapping approach also simulates the full predictive distribution, based on which the user can compute other uncertainty measures such as predictive intervals.

Only the Tweedie family of distributions are allowed, that is, the exponential family that admits a power variance function $V(\mu) = \mu^p$. The variance power p is specified in the `var.power` argument, and controls the type of the distribution. When the Tweedie compound Poisson distribution $1 < p < 2$ is to be used, the user has the option to specify `var.power = NULL`, where the variance power p will be estimated from the data using the `cplm` package [Zha12].

For example, the following fits the over-dispersed Poisson model and spells out the estimated reserve information:

```
R> # load data
R> data(GenIns)
R> GenIns <- GenIns / 1000
R> # fit Poisson GLM
R> (fit1 <- glmReserve(GenIns))

      Latest Dev.To.Date Ultimate  IBNR    S.E     CV
2       5339     0.98252     5434    95  110.1 1.1589
3       4909     0.91263     5379   470  216.0 0.4597
4       4588     0.86599     5298   710  260.9 0.3674
5       3873     0.79725     4858   985  303.6 0.3082
6       3692     0.72235     5111  1419  375.0 0.2643
7       3483     0.61527     5661  2178  495.4 0.2274
8       2864     0.42221     6784  3920  790.0 0.2015
```

```
9       1363     0.24162      5642  4279 1046.5 0.2446
10       344     0.06922      4970  4626 1980.1 0.4280
total  30457     0.61982     49138 18681 2945.7 0.1577
```

We can also extract the underlying GLM model by specifying `type = "model"` in the summary function:

```
R> summary(fit1, type = "model")

Call:
glm(formula = value ~ factor(origin) + factor(dev), family = fam,
    data = ldaFit, offset = offset)

Deviance Residuals:
    Min       1Q   Median       3Q      Max
-14.701   -3.913   -0.688    3.675   15.633

Coefficients:
                 Estimate Std. Error t value Pr(>|t|)
(Intercept)       5.59865    0.17292   32.38  < 2e-16
factor(origin)2   0.33127    0.15354    2.16   0.0377
factor(origin)3   0.32112    0.15772    2.04   0.0492
factor(origin)4   0.30596    0.16074    1.90   0.0650
factor(origin)5   0.21932    0.16797    1.31   0.1999
factor(origin)6   0.27008    0.17076    1.58   0.1225
factor(origin)7   0.37221    0.17445    2.13   0.0398
factor(origin)8   0.55333    0.18653    2.97   0.0053
factor(origin)9   0.36893    0.23918    1.54   0.1317
factor(origin)10  0.24203    0.42756    0.57   0.5749
factor(dev)2      0.91253    0.14885    6.13  4.7e-07
factor(dev)3      0.95883    0.15257    6.28  2.9e-07
factor(dev)4      1.02600    0.15688    6.54  1.3e-07
factor(dev)5      0.43528    0.18391    2.37   0.0234
factor(dev)6      0.08006    0.21477    0.37   0.7115
factor(dev)7     -0.00638    0.23829   -0.03   0.9788
factor(dev)8     -0.39445    0.31029   -1.27   0.2118
factor(dev)9      0.00938    0.32025    0.03   0.9768
factor(dev)10    -1.37991    0.89669   -1.54   0.1326

(Dispersion parameter for Tweedie family taken to be 52.6)

    Null deviance: 10699  on 54  degrees of freedom
Residual deviance:  1903  on 36  degrees of freedom
AIC: NA

Number of Fisher Scoring iterations: 4
```

Similarly, we can fit the Gamma and a compound Poisson GLM reserving model by changing the var.power argument:

```
R> # Gamma GLM
R> (fit2 <- glmReserve(GenIns, var.power = 2))

      Latest Dev.To.Date Ultimate  IBNR     S.E     CV
2       5339     0.98288     5432    93   45.17 0.4857
3       4909     0.91655     5356   447  160.56 0.3592
4       4588     0.88248     5199   611  177.62 0.2907
5       3873     0.79611     4865   992  254.47 0.2565
6       3692     0.71757     5145  1453  351.33 0.2418
7       3483     0.61440     5669  2186  526.29 0.2408
8       2864     0.43870     6529  3665  941.32 0.2568
9       1363     0.24854     5485  4122 1175.95 0.2853
10       344     0.07078     4860  4516 1667.39 0.3692
total  30457     0.62742    48543 18086 2702.71 0.1494

R> # compound Poisson GLM (variance function estimated from the data):
R> #(fit3 <- glmReserve(GenIns, var.power = NULL))
```

By default, the formulaic approach is used to compute the prediction errors. We can also carry out bootstrapping simulations by specifying mse.method = "bootstrap" (note that this argument supports partial match):

```
R> set.seed(11)
R> (fit5 <- glmReserve(GenIns, mse.method = "boot"))

      Latest Dev.To.Date Ultimate  IBNR    S.E     CV
2       5339     0.98252     5434    95  105.4 1.1098
3       4909     0.91263     5379   470  216.1 0.4597
4       4588     0.86599     5298   710  266.6 0.3755
5       3873     0.79725     4858   985  307.5 0.3122
6       3692     0.72235     5111  1419  376.3 0.2652
7       3483     0.61527     5661  2178  496.1 0.2278
8       2864     0.42221     6784  3920  812.9 0.2074
9       1363     0.24162     5642  4279 1050.9 0.2456
10       344     0.06922     4970  4626 2004.1 0.4332
total  30457     0.61982    49138 18681 2959.4 0.1584
```

When bootstrapping is used, the resulting object has three additional components - "sims.par", "sims.reserve.mean", and "sims.reserve.pred" that store the simulated parameters, mean values and predicted values of the reserves for each year, respectively.

```
R> names(fit5)

[1] "call"              "summary"           "Triangle"
[4] "FullTriangle"      "model"             "sims.par"
[7] "sims.reserve.mean" "sims.reserve.pred"
```

We can thus compute the quantiles of the predictions based on the simulated samples in the "sims.reserve.pred" element as:

```
R> pr <- as.data.frame(fit5$sims.reserve.pred)
R> qv <- c(0.025, 0.25, 0.5, 0.75, 0.975)
R> res.q <- t(apply(pr, 2, quantile, qv))
R> print(format(round(res.q), big.mark = ","), quote = FALSE)

   2.5%  25%   50%   75%   97.5%
2      0    34    82   170   376
3    136   337   470   615   987
4    279   556   719   917 1,302
5    506   797   972 1,197 1,674
6    774 1,159 1,404 1,666 2,203
7  1,329 1,877 2,210 2,547 3,303
8  2,523 3,463 3,991 4,572 5,713
9  2,364 3,593 4,310 5,013 6,531
10   913 3,354 4,487 5,774 9,165
```

The full predictive distribution of the simulated reserves for each year can be visualized easily:

```
R> library(ggplot2)
R> library(reshape2)
R> prm <- melt(pr)
R> names(prm) <- c("year", "reserve")
R> gg <- ggplot(prm, aes(reserve))
R> gg <- gg + geom_density(aes(fill = year), alpha = 0.3) +
+        facet_wrap(~year, nrow = 2, scales = "free")  +
+         theme(legend.position = "none")
R> print(gg)
```

4 Using ChainLadder with RExcel and SWord

The ChainLadder package comes with example files which demonstrate how its functions can be embedded in Excel and Word using the statconn interface[BN07].

The spreadsheet is located in the Excel folder of the package. The R command

```
R> system.file("Excel", package="ChainLadder")
```

will tell you the exact path to the directory. To use the spreadsheet you will need the RExcel-Add-in [BN07]. The package also provides an example SWord file, demonstrating how the functions of the package can be integrated into a MS Word file via SWord [BN07]. Again you find the Word file via the command:

```
R> system.file("SWord", package="ChainLadder")
```

The package comes with several demos to provide you with an overview of the package functionality, see

```
R> demo(package="ChainLadder")
```

5 Further resources

Other useful documents and resources to get started with R in the context of actuarial work:

- Introduction to R for Actuaries [DS06].
- An Actuarial Toolkit [MSH+06].
- *Computational Actuarial Science with R* [Ges14]
- *Modern Actuarial Risk Theory – Using R* [KGDD01]
- Actuar package vignettes: `http://cran.r-project.org/web/packages/actuar/index.html`
- Mailing list R-SIG-insurance[8]: Special Interest Group on using R in actuarial science and insurance

5.1 Other insurance related R packages

Below is a list of further R packages in the context of insurance. The list is by no-means complete, and the CRAN Task Views *'Emperical Finance'* and *Probability Distributions* will provide links to additional resources. Please feel free to contact us with items to be added to the list.

- `cplm`: Likelihood-based and Bayesian methods for fitting Tweedie compound Poisson linear models [Zha12].

[8] `https://stat.ethz.ch/mailman/listinfo/r-sig-insurance`

- `lossDev`: A Bayesian time series loss development model. Features include skewed-t distribution with time-varying scale parameter, Reversible Jump MCMC for determining the functional form of the consumption path, and a structural break in this path [LS11].

- `favir`: Formatted Actuarial Vignettes in R. FAViR lowers the learning curve of the R environment. It is a series of peer-reviewed Sweave papers that use a consistent style [Esc11].

- `actuar`: Loss distributions modelling, risk theory (including ruin theory), simulation of compound hierarchical models and credibility theory [DGP08].

- `fitdistrplus`: Help to fit of a parametric distribution to non-censored or censored data [DMPDD10].

- `mondate`: R packackge to keep track of dates in terms of months [Mur11].

- `lifecontingencies`: Package to perform actuarial evaluation of life contingencies [Spe11].

5.2 Presentations

Over the years the contributors of the `ChainLadder` package have given numerous presentations and most of those are still available online:

- Bayesian Hierarchical Models in Property-Casualty Insurance, Wayne Zhang, 2011

- ChainLadder at the Predictive Modelling Seminar, Institute of Actuaries, November 2010, Markus Gesmann, 2011

- Reserve variability calculations, CAS spring meeting, San Diego, Jimmy Curcio Jr., Markus Gesmann and Wayne Zhang, 2010

- The ChainLadder package, working with databases and MS Office interfaces, presentation at the "R you ready?" workshop , Institute of Actuaries, Markus Gesmann, 2009

- The ChainLadder package, London R user group meeting, Markus Gesmann, 2009

- Introduction to R, Loss Reserving with R, Stochastic Reserving and Modelling Seminar, Institute of Actuaries, Markus Gesmann, 2008

- Loss Reserving with R , CAS meeting, Vincent Goulet, Markus Gesmann and Daniel Murphy, 2008

- The ChainLadder package R-user conference Dortmund, Markus Gesmann, 2008

5.3 Further reading

Other papers and presentations which cited `ChainLadder` : [Orr07], [Nic09], [Zha10], [MNNV10], [Sch10], [MNV10], [Esc11], [Spe11]

6 Training and consultancy

Please contact us if you would like to discuss tailored training or consultancy.

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